Some Results on Local LR-degree Structures

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Definitions

$2^{<\omega}$: space of finite binary strings

2^{ω}: Cantor space; infinite binary strings

2^{ω} as a topological space: basic open sets are

$$[\sigma] := \{x \in 2^{\omega} : \sigma \subset x\}$$

2^{ω} as a measure space: Lebesgue measure given by

$$\mu\big([\sigma]\big) := 2^{-|\sigma|}$$

A *c.e. open set* is a set of finite strings $U \subset 2^{<\omega}$ such that:

- U is computably enumerable;
- if $\sigma, \tau \in U$ then $\sigma \not\subset \tau$ - the basic open sets $[\sigma], [\tau]$ are disjoint.

Also known as Σ^0_1-class.

Martin-Löf Randomness

x *is random if it is typical - has no distinguishing features*

A *test* is a sequence $(U_i)_{i\in\omega}$ of c.e. open sets such that $U_{i+1} \subseteq U_i$ and

$$\mu(U_i) \leq 2^{-i}.$$

$x \in 2^\omega$ is *random* if for all tests (U_i),

$$x \notin \bigcap U_i.$$

Theorem: There is a universal test $\tilde{U}_i$ such that

$$x \text{ is random iff } x \notin \cap \tilde{U}_i.$$

Relative randomness

These definitions relativise: add oracle A to tests to get A-randomness.

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$\rightarrow$ compare oracles A, B

by comparing their relative randomness A-rand, B-rand.

Definition

$A \leq_{LR} B$ iff $\forall x \in 2^\omega$,

$$x \text{ is B-random } \Rightarrow x \text{ is A-random.}$$

Equivalently, non-A-rand $\subseteq$ non-B-rand.

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Oracle B is at least as good at detecting patterns as oracle A.

- Equivalence relation: $A \equiv_{LR} B$ if $A \leq_{LR} B$ and $B \leq_{LR} A$

- LR-degrees are equivalence classes of sets under $\equiv_{LR}$

- $\mathbf{0}_{LR} = \deg_{LR} \emptyset$ consists of all low-for-random sets

Compared to Turing degrees:

- $A \leq_T B \Rightarrow A \leq_{LR} B$

- each LR-degree contains infinitely many Turing degrees

Theorem (Kjos-Hanssen)

$A \leq_{LR} B$ iff $\forall$ A-c.e. open sets U^A, $\mu(U^A) < 1$,

$$(*) \qquad U^A \subseteq V^B$$

for some B-c.e. open set V^B with $\mu(V^B) < 1$.

In fact, $(*)$ need hold only for a single U

- member of universal A-randomness test.

$\rightarrow$ gives a means of *creating* and *destroying* LR-reductions.

Working with LR-incomplete sets

We can use the fact that $\emptyset' \not\leq_{LR} A$ to impose restraint.

If $\emptyset' \not\leq_{LR} A$ then

$$U^{\emptyset'} \subseteq F^A \Rightarrow \mu(F^A) = 1.$$

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    $\rightarrow \emptyset' \not\leq_{LR} A$ guarantees sufficiently many will succeed to force

$$\mu(F^A) = 1.$$

Some applications

An alternate proof of

Theorem (Barmpalias, Lewis, Stephan (2008))

Given a c.e. set $A <_{LR} \emptyset'$, *there is a c.e. set* B *such that*

$$A <_{LR} B <_{LR} \emptyset'.$$

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Can the theorem be extended to any Δ^0_2 $A <_{LR} \emptyset'$?

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Given a Δ^0_2 set $\emptyset <_{LR} A <_{LR} \emptyset'$, there is (uniformly) a c.e. set B LR-incomparable with A.

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Full approximation proof

Sketch of proof (c.e. case)

Given c.e. set A, construct c.e. B to satisfy

$$P_e \; : \qquad T_e^B \not\subseteq V_e^A$$

$$N_e \; : \qquad U^A \not\subseteq {V_e}^A$$

$(V_e)_e$ is a listing of all LR-reductions: Σ_1^0 operators s.t. $\mu(V_e^X) < 1$.

N requirements

Sacks restraints adapted to LR reductions

As in Barmpalias, Lewis, Soskova (2008)

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- removing σ from T^B via B-change and
- (attempting to) restrain A on use of $\sigma \subseteq V_e^A$.

$\mu(V_e^A)$ increases by $\mu(\sigma)$, $\mu(T^B)$ does not.

After finitely many repetitions $\mu(V_e^A)$ cannot increase further.

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- when $\sigma \subseteq V_e^A[s]$: *try to restrain* A
  - $+$ put ρ into F^A with same use
  - $+$ wait for $\emptyset'$-change

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  $\rightarrow$ if $\emptyset'$ never changes, $\sigma \in T^B$ permanently

  $\rightarrow$ can happen for at most $\mu(U^{\emptyset'})$ many σ.
- if computation $\sigma \subseteq V_e^A[s]$ is later destroyed by A-change: attack σ fails

  $\rightarrow$ each $\rho \in F^A$ corresponds to successful $\sigma \subseteq V_e^A$

  $\rightarrow$ $A <_{LR} \emptyset'$ guarantees sufficiently many will succeed.

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