Non-cupping and non-mitoticity in the LR-degrees

Anthony Morphett

Leeds Logic Seminar
10 October 2007

Outline

- LR-reducibility and LR-degrees
- Non-cupping and LR-completeness
- Mitoticity and non-mitoticity

Algorithmic Randomness

Consider infinite binary strings $x = x_0 x_1 x_2 \ldots$, $x_i \in \{0, 1\}$.

When is such a string *random*?

Three ideas:

Algorithmic Randomness

Consider infinite binary strings $x = x_0 x_1 x_2 \ldots$, $x_i \in \{0, 1\}$.

When is such a string *random*?

Three ideas:

• x should be unpredictable: gambler should not win if betting on consecutive bits

Algorithmic Randomness

Consider infinite binary strings $x = x_0 x_1 x_2 \ldots$, $x_i \in \{0, 1\}$.

When is such a string *random*?

Three ideas:

• x should be unpredictable: gambler should not win if betting on consecutive bits

• x should be typical: should not have any distinguishing properties

Algorithmic Randomness

Consider infinite binary strings $x = x_0 x_1 x_2 \ldots$, $x_i \in \{0, 1\}$.

When is such a string *random*?

Three ideas:

• x should be unpredictable: gambler should not win if betting on consecutive bits

• x should be typical: should not have any distinguishing properties

• x should be incompressible: no way to describe (initial segments of) x except by x itself

Definitions

$2^{<\omega}$: space of finite binary strings

2^{ω}: Cantor space; infinite binary strings

2^{ω} as a topological space: basic open sets are

$$[\sigma] := \{x \in 2^{\omega} : \sigma \subset x\}$$

2^{ω} as a measure space: Lebesgue measure given by

$$\mu\big([\sigma]\big) := 2^{-|\sigma|}$$

A *c.e. open set* is a set of finite strings $U \subset 2^{<\omega}$ such that:

- U is computably enumerable;
- if $\sigma, \tau \in U$ then $\sigma \not\subset \tau$ - the basic open sets $[\sigma], [\tau]$ are disjoint.

Also known as Σ^0_1-class.

Martin-Löf Randomness

x *is random if it is typical - has no distinguishing features*

A *test* is a sequence $(U_i)_{i\in\omega}$ of c.e. open sets such that $U_{i+1} \subseteq U_i$ and

$$\mu(U_i) \leq 2^{-i}.$$

$x \in 2^\omega$ is *random* if for all tests (U_i),

$$x \notin \bigcap U_i.$$

Theorem: There is a universal test $\tilde{U}_i$ such that

$$x \text{ is random iff } x \notin \cap \tilde{U}_i.$$

x is random if it is unpredictable - a gambler should not win if betting on consecutive bits

- no c.e. martingale (gambling strategy) succeeds on x in the limit

x is random if it is incompressible - there is no way to describe (initial segments of) x except by x itself

- the shortest Turing program that outputs $x \upharpoonright n$ is as long as $x \upharpoonright n$
- $x \upharpoonright n$ is hard-coded into the program
- high Kolmogorov complexity

Low for randomness

These definitions relativise: add oracle A to tests to get A-randomness.

$$x \text{ is } A\text{-random if } x \notin \bigcap U_i^A \text{ for universal oracle test } U_i.$$

A is *low for random* if

$$x \text{ is random } \Rightarrow x \text{ is } A\text{-random}$$

"Everything random is still A-random" - using A as oracle doesn't help detect patterns.

Low for randomness

- Computable sets are low for random
- Non-computable low for random sets exist
- All low-for-randoms are Δ^0_2 and low

LR-reducibility

Low-for-randomness suggests the definition of LR-reducibility:

Definition

$A \leq_{LR} B$ iff $\forall x \in 2^{\omega}$,

$$x \text{ is } B\text{-random } \Rightarrow x \text{ is } A\text{-random}.$$

A cannot detect any more patterns than B.

- Equivalence relation: $A \equiv_{LR} B$ if $A \leq_{LR} B$ and $B \leq_{LR} A$
- LR-degrees are equivalence classes of sets under $\equiv_{LR}$
- $\mathbf{0}_{LR} = \deg_{LR} \emptyset$ consists of all low-for-random sets

Compared to Turing degrees:

- $A \leq_T B \Rightarrow A \leq_{LR} B$
- each LR-degree contains infinitely many Turing degrees

Theorem (Kjos-Hanssen)

$A \leq_{LR} B$ *iff* $\forall$ A*-c.e. open sets* U^A, $\mu(U^A) < 1$,

$$(*) \qquad U^A \subseteq V^B$$

for some B*-c.e. open set* V^B *with* $\mu(V^B) < 1$.

In fact, $(*)$ need hold only for a single U

- member of universal A-randomness test.

$\rightarrow$ gives a means of *creating* and *destroying* LR-reductions.

Non-cupping and LR-degrees

A c.e. set A is *non-cuppable* if $\forall$ c.e. sets X,

$$A \oplus X \geq_T \emptyset' \Rightarrow X \geq_T \emptyset'$$

- you can't get to $\emptyset'$ by adding the information of another c.e. set except $\emptyset'$ itself

NCup $= \{$ non-cuppable c.e. Turing degrees $\}$

Non-cupping in Turing degrees studied by

- Yates, Cooper 1972
- Harrington (D. Miller), 1970's, 80's
- Li, Slaman & Yang; Yang & Yu; tree constructions

Non-cupping in Turing degrees studied by

- Yates, Cooper 1972
- Harrington (D. Miller), 1970's, 80's
- Li, Slaman & Yang; Yang & Yu; tree constructions

NCup forms an ideal in c.e. Turing degrees:

- closed under $\oplus$
- closed downward under $\leq_T$

Theorem (Cooper, Yates)

There is a nontrivial non-cuppable c.e. degree.

Theorem (Harrington)

1. *There is a high non-cuppable c.e. degree.*
2. *Moreover, for any high* $\mathbf{b}$ *there is a high* $\mathbf{a}$ *such that* $\mathbf{a}$ *cannot be cupped to* $\mathbf{b}$*:*

$$\forall \mathbf{x} \ \ \mathbf{a} \cup \mathbf{x} \geq \mathbf{b} \Rightarrow \mathbf{x} \geq \mathbf{b}.$$

Theorem

There is a non-cuppable c.e. set $A \equiv_{LR} \emptyset'$ *(LR-complete).*

Theorem

There is a non-cuppable c.e. set $A \equiv_{LR} \emptyset'$ *(LR-complete).*

$A \geq_{LR} \emptyset' \Rightarrow A$ is high...

Theorem

There is a non-cuppable c.e. set $A \equiv_{LR} \emptyset'$ *(LR-complete).*

$A \geq_{LR} \emptyset' \Rightarrow A$ is high...

so this is a partial strengthening of Harrington's result.

Capping and non-cupping

Cap $=$ cappable degrees, half of minimal pair

Capping and non-cupping

Cap = cappable degrees, half of minimal pair

Theorem (Barmpalias & Montalbán)

There is a cappable LR-complete c.e. set.

Capping and non-cupping

Cap = cappable degrees, half of minimal pair

Theorem (Barmpalias & Montalbán)

There is a cappable LR-complete c.e. set.

NCup is a subideal of **Cap**

The construction

To make $A \geq_{LR} \emptyset'$:

- if σ appears in $U^{\emptyset'}$ then put it in V^A with large use u
- if σ is removed from $U^{\emptyset'}$ due to $\emptyset'$-change, put u into A
- this may remove some other legitimate interval ρ with use $> u$; put ρ back into V^A with the same use.

Making A non-cuppable

To make A non-cuppable we would like to build Turing functional Δ to satisfy

$$N_e : \Gamma^{A \oplus W} = \emptyset' \;\Rightarrow\; \Delta^{W} = \emptyset'$$

for all Turing functionals Γ and c.e. sets W.

Idea:

- Wait until $\Gamma^{AW}(p)\downarrow = \emptyset'(p)$;
- define $\Delta^{W}(p) = \Gamma^{AW}(p)$;
- restrain $A \restriction use \Gamma^{AW}(p)$.

Non-cupping strategy - naive

Problems:

1. if in fact $\Gamma^{AW} = \emptyset'$, we must act infinitely often

   $\Rightarrow$ N_e imposes infinite restraint

   $\Rightarrow$ must spread actions over infinitely many subrequirements

$$M_{e,p} : \Gamma^{AW}(p) = \emptyset'(p) \Rightarrow \Delta^{W}(p)\downarrow = \emptyset'(p)$$

2. need to be able to invalidate $\Delta^{W}(p)$ definitions to right of current path
   - must maintain A-restraint while $\Delta^{W}(p)$ is defined
   - need a way to force W-change

Non-cupping strategy - improved

We build auxiliary c.e. set D. Let

$$K = D \cup \emptyset' \qquad (\equiv_T \emptyset')$$

N Parent node: τ

- waits for expansionary stage for $\Gamma^{AW} = K$

M_p Subrequirement node: α

- chooses *flip-point* $d \notin D$
- waits until $\Gamma^{AW}(d)\downarrow$
- defines $\Delta^W(p)\downarrow = \Gamma^{AW}(p) = \emptyset'(p)$ with use $u = \mathit{use}\,\Gamma^{AW}(d)$

If we need to invalidate α's $\Delta^{W}(p)$ definition:

- enumerate d into D
- K changes, so $\Gamma^{AW} = K$ is destroyed
- if $\Gamma^{AW} = K$ then Γ^{AW} must change to restore agreement with K
- but A is restrained, so W must change below

$$\mathit{use}\,\Gamma^{AW}(d) = \mathit{use}\,\Delta^{W}(p)$$

- previous definition $\Delta^{\sigma}(p)$ is invalidated as now $\sigma \not\subset W$

Putting them together - non-cuppable and LR-complete

- Restraints by non-cupping requirements prevent us from removing intervals from V^A
- Give each requirement a quota ϵ
- Allow it to capture at most ϵ junk intervals
- Choose ϵ's so that

$$\sum \epsilon < \frac{1}{2}$$

Thus

$$\mu(V^A) \ < \ \mu(U^{\emptyset'}) + \sum \epsilon \ < \ 1.$$

In tree setting, this means:

- allowing only one restraint on each level of the tree
- providing non-cupping requirements with an estimate to $\mu\left(U^{\emptyset'}\right)$
- resetting nodes if their measure estimate is wrong

Notable features of the construction

Regarding the LR-complete strategy:

- Uses measure-guessing backup strategies as in previous LR-complete constructions
- Can't always reset a node when its measure guess is wrong
  - use non-cupping clearing procedure instead
- Permanent restraints can capture more than their quota ϵ of junk intervals
- But still ensure that

$$\sum_{M_p} \epsilon(M_p) \; < \; 3\,\epsilon(N)$$

Notable features of the construction

Regarding the non-cuppable strategy:

- Must delay the definition of $\Delta^W(p)$ until

$$\mu\big(V^{A\upharpoonright u} - V^{A\upharpoonright R} - U^{\emptyset'}\big) < \epsilon$$

  That is, until we won't capture more than ϵ junk.

- Must clear definitions by nodes to the left, as well as above, before visiting a node

Mitoticity and LR-degrees

Definition

A c.e. set A is *mitotic* if there are disjoint c.e. sets X, Y s.t.

$$A = X \cup Y; \quad \text{and} \quad X \equiv_T Y \equiv_T A.$$

"A can be split into two c.e. sets of the same degree"

Mitoticity and LR-degrees

Definition

A c.e. set A is *mitotic* if there are disjoint c.e. sets X, Y s.t.

$$A = X \cup Y; \quad \text{and} \quad X \equiv_T Y \equiv_T A.$$

"A can be split into two c.e. sets of the same degree"

- there are non-mitotic c.e. sets

Mitoticity and LR-degrees

Definition

A c.e. set A is *mitotic* if there are disjoint c.e. sets X, Y s.t.

$$A = X \cup Y; \quad \text{and} \quad X \equiv_T Y \equiv_T A.$$

"A can be split into two c.e. sets of the same degree"

- there are non-mitotic c.e. sets
- they can be complete ($\equiv_T \emptyset'$)

Mitoticity and LR-degrees

Definition

A c.e. set A is *mitotic* if there are disjoint c.e. sets X, Y s.t.

$$A = X \cup Y; \quad \text{and} \quad X \equiv_T Y \equiv_T A.$$

"A can be split into two c.e. sets of the same degree"

- there are non-mitotic c.e. sets
- they can be complete ($\equiv_T \emptyset'$)
- for every c.e. set B there is a non-mitotic c.e. set $A \leq_T B$

Mitoticity and LR-degrees

Definition

A c.e. set A is *mitotic* if there are disjoint c.e. sets X, Y s.t.

$$A = X \cup Y; \quad \text{and} \quad X \equiv_T Y \equiv_T A.$$

"A can be split into two c.e. sets of the same degree"

- there are non-mitotic c.e. sets
- they can be complete ($\equiv_T \emptyset'$)
- for every c.e. set B there is a non-mitotic c.e. set $A \leq_T B$
- there is a completely mitotic c.e. Turing degree $\mathbf{d}$:
  - every c.e. $A \in \mathbf{d}$ is mitotic

[Ladner 1973 & others]

Theorem

- *There is a non-LR-mitotic c.e. set* $A \equiv_T \emptyset'$;
- *For every non-low-for-random c.e. set* B *there is a c.e. set* $A \leq_T B$ *that cannot be split into two c.e. sets of the same LR-degree.*

Basic strategy

Build c.e. A in stages; also build A-c.e. open set T

For a possible splitting X, Y and LR-reduction V, q:

Basic strategy

Build c.e. A in stages; also build A-c.e. open set T

For a possible splitting X, Y and LR-reduction V, q:

- put some interval σ into T^A with use k

Basic strategy

Build c.e. A in stages; also build A-c.e. open set T

For a possible splitting X, Y and LR-reduction V, q:

- put some interval σ into T^A with use k
- wait until

$$X \cup Y[s] = A[s] \text{ and } X[s] \cap Y[s] = \emptyset \tag{1}$$

and

$$\sigma \subset V^X, \sigma \subset V^Y \tag{2}$$

Basic strategy

Build c.e. A in stages; also build A-c.e. open set T

For a possible splitting X, Y and LR-reduction V, q:

- put some interval σ into T^A with use k
- wait until

$$X \cup Y[s] = A[s] \text{ and } X[s] \cap Y[s] = \emptyset \tag{1}$$

and

$$\sigma \subset V^X, \sigma \subset V^Y \tag{2}$$

- put k into A
  - this removes σ from T^A

Basic strategy

Build c.e. A in stages; also build A-c.e. open set T

For a possible splitting X, Y and LR-reduction V, q:

- put some interval σ into T^A with use k
- wait until

$$X \cup Y[s] = A[s] \text{ and } X[s] \cap Y[s] = \emptyset \tag{1}$$

and

$$\sigma \subset V^X, \sigma \subset V^Y \tag{2}$$

- put k into A
  - this removes σ from T^A
- restrain $A \restriction u$
  - u = use of σ in V^X, V^Y

Outcome:

$A \upharpoonright u$ is fixed except for k

Outcome:

$A{\upharpoonright}u$ is fixed except for k

$\rightarrow$ $X{\upharpoonright}u$, $Y{\upharpoonright}u$ is fixed except for k

Outcome:

$A \restriction u$ is fixed except for k

$\rightarrow$ $X \restriction u$, $Y \restriction u$ is fixed except for k

$\rightarrow$ only one of $X \restriction u$, $Y \restriction u$ can change if X, Y is a splitting of A

Outcome:

$A{\restriction}u$ is fixed except for k

$\rightarrow$ $X{\restriction}u, Y{\restriction}u$ is fixed except for k

$\rightarrow$ only one of $X{\restriction}u, Y{\restriction}u$ can change if X, Y is a splitting of A

$\rightarrow$ one of V^X, V^Y retains σ.

Outcome:

$A\restriction u$ is fixed except for k

$\rightarrow$ $X\restriction u$, $Y\restriction u$ is fixed except for k

$\rightarrow$ only one of $X\restriction u$, $Y\restriction u$ can change if X, Y is a splitting of A

$\rightarrow$ one of V^X, V^Y retains σ.

Net result: force one of $\mu\big(V^X\big), \mu\big(V^Y\big)$ to increase by $\epsilon = \mu(\sigma)$

Outcome:

$A \upharpoonright u$ is fixed except for k

$\rightarrow$ $X \upharpoonright u$, $Y \upharpoonright u$ is fixed except for k

$\rightarrow$ only one of $X \upharpoonright u$, $Y \upharpoonright u$ can change if X, Y is a splitting of A

$\rightarrow$ one of V^X, V^Y retains σ.

Net result: force one of $\mu(V^X), \mu(V^Y)$ to increase by $\epsilon = \mu(\sigma)$

Do this q/ϵ many times and we force $\mu(V^X), \mu(V^Y) > q$.

Outcome:

$A\restriction u$ is fixed except for k

$\rightarrow$ $X\restriction u$, $Y\restriction u$ is fixed except for k

$\rightarrow$ only one of $X\restriction u$, $Y\restriction u$ can change if X, Y is a splitting of A

$\rightarrow$ one of V^X, V^Y retains σ.

Net result: force one of $\mu\big(V^X\big), \mu\big(V^Y\big)$ to increase by $\epsilon = \mu(\sigma)$

Do this q/ϵ many times and we force $\mu\big(V^X\big), \mu\big(V^Y\big) > q$.

Combine the requirements for each possible splitting (X, Y, V, q) in a finite injury construction.

Permitting

Given non-computable c.e. B, construct $A \leq_T B$ by permitting:

only change $A \restriction u$ if $B \restriction u$ changes also.

Permitting

Given non-computable c.e. B, construct $A \leq_T B$ by permitting:

only change $A{\upharpoonright}u$ if $B{\upharpoonright}u$ changes also.

- wait for suitable B-change before removing σ from T^A

Permitting

Given non-computable c.e. B, construct $A \leq_T B$ by permitting:

only change $A\restriction u$ if $B\restriction u$ changes also.

• wait for suitable B-change before removing σ from T^A

• permission may not occur; must use many σ's to ensure that enough succeed

Non-low-for-random permitting

Must ensure we get q/ϵ many B-changes for each requirement.

Idea: if B is not low for random, then $B \not\leq_{LR} \emptyset$, so

$$U_i^B \subset E \Rightarrow \mu(E) = 1$$

for universal test member U_i, c.e. open set E

Non-low-for-random permitting

Must ensure we get q/ϵ many B-changes for each requirement.

Idea: if B is not low for random, then $B \not\leq_{LR} \emptyset$, so

$$U_i^B \subset E \Rightarrow \mu(E) = 1$$

for universal test member U_i, c.e. open set E

If we try to trace U_i^B into E, guaranteed to get enough B-changes to force $\mu(E) = 1$.

Choose i s.t.

$$\mu(U_i^X) \leq \epsilon$$

Non-low-for-random permitting

Must ensure we get q/ϵ many B-changes for each requirement.

Idea: if B is not low for random, then $B \not\leq_{LR} \emptyset$, so

$$U_i{}^B \subset E \Rightarrow \mu(E) = 1$$

for universal test member U_i, c.e. open set E

If we try to trace $U_i{}^B$ into E, guaranteed to get enough B-changes to force $\mu(E) = 1$.

Choose i s.t.

$$\mu(U_i{}^X) \leq \epsilon$$

Before putting σ in T^A, choose $\rho \in U_i{}^B$

Non-low-for-random permitting

Must ensure we get q/ϵ many B-changes for each requirement.

Idea: if B is not low for random, then $B \not\leq_{LR} \emptyset$, so

$$U_i{}^B \subset E \Rightarrow \mu(E) = 1$$

for universal test member U_i, c.e. open set E

If we try to trace $U_i{}^B$ into E, guaranteed to get enough B-changes to force $\mu(E) = 1$.

Choose i s.t.

$$\mu(U_i{}^X) \leq \epsilon$$

Before putting σ in T^A, choose $\rho \in U_i{}^B$

When ready to remove σ, put ρ in E; wait for B-change removing ρ

if never, then $\rho \in U_i{}^B$ - true only for $\mu(U_i{}^B) < \epsilon$ many ρ's.

Questions

Mitoticity:

- Is there a completely LR-mitotic c.e. LR-degree?

Questions

Mitoticity:

- Is there a completely LR-mitotic c.e. LR-degree?

Non-cupping:

- Make non-cuppable A even higher: ultrahigh
- Harrington's theorem for LR-completeness