INCOMPLETENESS, APPROXIMATION AND RELATIVE RANDOMNESS

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Abstract. We present some results about the structure of c.e. and Δ^0_2 LR-degrees. First we give a technique for lower cone avoidance in the c.e. and Δ^0_2 LR-degrees, and combine this with upper cone avoidance via Sacks restraints to construct a c.e. LR-degree which is incomparable with a given intermediate Δ^0_2 LR-degree. Next we combine measure-guessing with an LR-incompleteness strategy to construct an incomplete c.e. LR-degree which is above a given low Δ^0_2 LR-degree. This is in contrast to the Turing degrees, in which there is a low Δ^0_2 Turing degree which is incomparable with all intermediate c.e. Turing degrees.

1. Introduction

A basic task of computability theory is to study the information content of a set X by studying classes of computations obtained by relativising to X. The most fundamental class of relativised computations is the partial recursive functions relative to X, $\mathrm{Rec}(X)$. We can compare the information content of two sets $A, B \subseteq \mathbb{N}$ by comparing the classes $\mathrm{Rec}(A)$ and $\mathrm{Rec}(B)$. This approach gives rise to the familiar Turing reducibility, which may be defined as

$$A \leq_T B \text{ iff } \mathrm{Rec}(A) \subseteq \mathrm{Rec}(B). \tag{1}$$

By replacing the class $\mathrm{Rec}(X)$ with another class of computations, we can obtain other reducibilities. One class of computations that has been of particular interest recently due to the study of algorithmic randomness is the class of effective approximations of nullsets (measure 0 subsets) of Cantor space. Effective approximations of nullsets are called Martin-Löf tests, and give rise to the notion of Martin-Löf randomness (defined formally below). One may study the information content of a set A by examining the class of nullsets that are effectively approximable relative to the oracle A, or equivalently, the notion of Martin-Löf randomness relativised to A. We obtain a reducibility, known as LR-reducibility, by comparing the contents of effectively approximable nullsets: $A \leq_{LR} B$ iff every nullset that is effectively approximable relative to A is contained in a nullset that is effectively approximable relative to B. LR-reducibility was first considered in [16], and has been further studied, for instance, in [3] and [4].

LR-reducibility has been shown to be equivalent to another reducibility arising from algorithmic randomness, known as LK-reducibility. For $A, B \subseteq \mathbb{N}$, define

$$A \leq_{LK} B \text{ iff } \exists c \in \mathbb{N}\, \forall \sigma \in 2^{<\omega}\; K^B(\sigma) \leq K^A(\sigma) + c$$

where $K^X(\sigma)$ is the prefix-free Kolmogorov complexity of σ relative to the oracle X. Informally, this asserts that any string that can be compressed using the oracle A can also be compressed at least as well using the oracle B. It was shown in [10] that the relations $\leq_{LK}$ and $\leq_{LR}$ coincide. A proof may be found in [15].

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The reducibility $\leq_{LR}$ ($\leq_{LK}$) has several similarities with $\leq_T$. As a preordering, $\leq_{LR}$ is a proper weakening of $\leq_T$[1], and both $\leq_T$ and $\leq_{LR}$ are Σ^0_3 relations[2]. There are however some notable differences between $\leq_{LR}$ and $\leq_T$. Of particular note is the existence of uncountable lower cones in the $\leq_{LR}$ preordering: there are sets A such that $\{X : X \leq_{LR} A\}$ is uncountable [3, 13]. Work in [12] and [2] has characterised such sets as the non-low-for-Ω sets.

Focussing our attention to the LR-degrees of c.e. and Δ^0_2 sets, many techniques familiar from the study of the c.e. and Δ^0_2 Turing degrees can be adapted to work, with suitable modifications, in the context of LR-reducibility. For example, Barmpalias, Lewis and Stephan [4] used a finite-extension technique to prove that for every intermediate Δ^0_2 LR-degree there is an incomparable Δ^0_2 LR-degree. That is, if A is Δ^0_2 and $\emptyset <_{LR} A <_{LR} \emptyset'$ then

$$\exists\ \Delta^0_2 \text{ set } B \text{ with } A \mid_{LR} B. \tag{2}$$

By relativising an earlier splitting theorem from [3], Barmpalias [1] established the upward density of the Δ^0_2 LR-degrees; that is, if A is Δ^0_2 and $A <_{LR} \emptyset'$, then

$$\exists\ \Delta^0_2 \text{ set } B \text{ with } A <_{LR} B <_{LR} \emptyset'. \tag{3}$$

Barmpalias also developed a permitting technique to show that every non-zero Δ^0_2 LR-degree bounds a non-zero c.e. LR-degree: if A is Δ^0_2 and $A \not\leq_{LR} \emptyset$ then

$$\exists \text{ c.e. set } B \text{ with } \emptyset <_{LR} B <_{LR} A. \tag{4}$$

Moreover, Barmpalias showed that his technique can be applied to two Δ^0_2 sets simultaneously, establishing that every pair of non-zero Δ^0_2 LR-degrees bounds a c.e. LR-degree. This gives an elementary difference between the structures of Δ^0_2 LR-degrees and the Δ^0_2 Turing degrees. These latter results show some notable contrasts between the c.e. LR-degrees and c.e. Turing degrees within the Δ^0_2 LR-degrees and Δ^0_2 Turing degrees, respectively.

In this paper, we continue the study of the c.e. LR-degrees within the Δ^0_2 LR-degrees. In section 4 we prove (Theorem 2) that for every intermediate Δ^0_2 LR-degree there is an incomparable c.e. LR-degree. This is a strengthening of the earlier result of [4], establishing that the set B in (2) can be made c.e. The construction uses a technique for leveraging the LR-incompleteness of a Δ^0_2 set that is outlined in section 3, analogous to the coding strategy first used by Sacks [18] to prove a similar result in the c.e. Turing degrees.

In section 5 we show (Theorem 14) that every low Δ^0_2 LR-degree is bounded by an incomplete c.e. LR-degree. This is in contrast to the situation in the Δ^0_2 Turing degrees, in which there is a low degree that is incomparable with all intermediate c.e. degrees. Theorem 14 is a partial strengthening of Barmpalias's result (3); it shows that the set B can be made c.e. in the case that A is low. It is unknown if the requirement that A is low can be weakened. Theorem 14 can also be seen as a partial dual of (4).

Note. Diamondstone [8] has recently proved the stronger result that any pair of low Δ^0_2 LR-degrees are bounded by a low c.e. LR-degree. Theorem 14 follows directly from Diamondstone's result. Diamondstone's construction uses a different technique to that used in section 5, and appeared after the work presented in this paper was done.

2. Preliminaries

We call members of $2^{<\omega}$ *strings*, and members of 2^ω *reals*. We identify reals with subsets of $\mathbb{N}$ in the usual way. For $X \in 2^\omega$ and $n \in \mathbb{N}$, $X \upharpoonright n$ denotes the initial segment of X of length n. For $\sigma, \tau \in 2^{<\omega}$ and $X \in 2^\omega$, we write $\sigma \subseteq \tau$ and $\sigma \subset X$ to denote that σ is an initial segment of

[1] The implication $A \leq_T B \Rightarrow A \leq_{LR} B$ is immediate, and the failure of the converse implication follows from the existence of a non-computable low-for-random set [11].

[2] $\leq_{LR}$ can be seen to be Σ^0_3, for instance, by Theorem 1.

τ and X, respectively. The length of a string is denoted $|\sigma|$. We obtain the standard bijection between $2^{<\omega}$ and $\mathbb{N}$ by ordering the finite strings first by length and then lexicographically.

The basic clopen sets of Cantor space 2^ω are of the form $[\sigma] = \{X \in 2^\omega : \sigma \subset X\}$ for $\sigma \in 2^{<\omega}$. To simplify presentation we will often omit the brackets and denote by σ both the string and the clopen set (and similarly for sets of strings). It will be clear from context which is intended. The Lebesgue measure on 2^ω is denoted by μ.

An oracle Σ^0_1 class V is a procedure for uniformly producing an A-c.e. set of strings V^A (or the A-Σ^0_1 class $[V^A]$) from an oracle A. Such an operator can be considered as a c.e. set of axioms of the form (σ, τ) for $\sigma, \tau \in 2^{<\omega}$ which assert that $[\tau] \subseteq [V^A]$ if $\sigma \subset A$. An oracle Σ^0_1 class V is *bounded* if there is a $p \in \mathbb{Q}, 0 < p < 1$ such that

$$\mu(V^X) \leq p \text{ for all } X \in 2^\omega.$$

An oracle Martin-Löf test is a sequence $(U_i)_{i\in\mathbb{N}}$ of oracle Σ^0_1 classes (uniform in i), such that

$$\mu(U_i^X) \leq 2^{-i} \text{ for all } X \in 2^\omega.$$

A real X is *Martin-Löf random relative to* A (or A-random for short) if $X \notin \bigcap_i U_i^A$ for any oracle Martin-Löf test $(U_i)_{i\in\mathbb{N}}$. An important feature of Martin-Löf randomness is the existence of a universal test. Let $(U_{k,i})_{i\in\mathbb{N}}$ for $k \in \mathbb{N}$ be a computable listing of all oracle Martin-Löf tests; that is, $U_{k,i}$ is the i'th member of the k'th test. Then

$$U_i^X := \bigcup_k U^X_{k,i+k+1}$$

for $i \in \mathbb{N}$ gives a universal Martin-Löf test: X is A-random iff $X \notin \bigcap_i U_i^A$. We denote the class of reals that are A-random by $\mathsf{MLR}(A)$.

Equipped with the notion of Martin-Löf randomness, we can compare the information content of oracles A, B by comparing the notions of randomness obtained by relativising to A and B. A natural way to do this is LR-reducibility. For $A, B \in 2^\omega$, we say that

$$A \leq_{LR} B \text{ iff } \mathsf{MLR}(B) \subseteq \mathsf{MLR}(A).$$

This can be expressed perhaps more intuitively by analogy with (1) by stating:

$$A \leq_{LR} B \text{ iff } \bigcap_i U_i^A \subseteq \bigcap_i U_i^B$$

where $(U_i)_{i\in\mathbb{N}}$ is the universal oracle Martin-Löf test defined above. That is, $A \leq_{LR} B$ if the universal nullset computed by A is contained in the universal nullset computed by B. We obtain the LR-degrees by defining $A \equiv_{LR} B$ if both $A \leq_{LR} B$ and $B \leq_{LR} A$. An *LR-degree* is an equivalence class under $\equiv_{LR}$. The LR-degree of $\emptyset$ consists of the *low-for-random* sets. Such sets were first considered in [11], where a non-computable c.e. low-for-random set was constructed. The notion of low-for-randomness has since been extensively studied, for instance in [16], where the property of being low-for-random was shown to coincide with several other properties including K-triviality.

An important tool for working with LR-reducibility is the following theorem of Kjos-Hanssen [9]. A proof may be found in [15].

Theorem 1. Let $A, B \in 2^\omega$. The following are equivalent.

(1) $A \leq_{LR} B$;
(2) for every bounded A-Σ^0_1 class T^A there is a bounded B-Σ^0_1 class V^B with $T^A \subseteq V^B$;
(3) for some member U of the universal oracle Martin-Löf test, there is a bounded B-Σ^0_1 class V^B with $U^A \subseteq V^B$.

3. Working with an LR-incomplete set

In this section we outline a technique for working with an LR-incomplete c.e. or Δ^0_2 set in full-approximation constructions. We will use this technique as part of a lower cone avoidance strategy in Theorem 2. The technique is a method for leveraging the LR-incompleteness of a set A to limit the changes in the approximation of A, effectively imposing 'restraints' on A which can be utilised by other requirements in a construction.

Suppose that the c.e. set A is LR-incomplete and F^A is an A-Σ^0_1 class; then by Theorem 1 we have

$$U^{\emptyset'} \subseteq F^A \Rightarrow \mu(F^A) = 1$$

for a member U of a universal oracle Martin-Löf test. If we attempt to trace $U^{\emptyset'}$ into an A-Σ^0_1 class F^A, then we are guaranteed that $\emptyset'$ will change more frequently than A, frequently enough to ensure $\mu(F^A) = 1$. We can use this to our advantage to provide restraints on A. Suppose that during a construction we wish to restrain $A \upharpoonright u$ at stage s. We can take a string ρ from $U^{\emptyset'}[s]$ which is not yet in F^A, and enumerate ρ into F^A with use u. Then we wait for a $\emptyset'$-change below the use of the computation $\rho \in U^{\emptyset'}[s]$. If the $\emptyset'$-change never occurs, then we never proceed further with this attempt, and the restraint is unsuccessful; we say that the attempt is *stalled*. However in this case we have that $\rho \in U^{\emptyset'}$ permanently; this can happen for at most $\mu(U^{\emptyset'})$ worth of strings ρ, and this can be made as small as necessary by choosing a suitably small U. Otherwise, a $\emptyset'$-change eventually occurs. At this point, we have $\rho \in F^A$ but $\rho \notin U^{\emptyset'}$. If A later changes below u, then the attempt at restraining $A \upharpoonright u$ is unsuccessful. However we can argue that sufficiently many attempts will be successful (ie A will not change below u after the $\emptyset'$-change) to ensure that $\mu(F^A) = 1$.

If our requirement is such that it requires a finite measure worth of restraints for satisfaction, then we can argue that it will be satisfied with the above method. Suppose that it is not satisfied. We will make infinitely many attempts at restraining $A \upharpoonright u$ for some u, each attempt corresponding to a string from $U^{\emptyset'}[s]$. Infinitely many will correspond to the true strings of $U^{\emptyset'}$. Since a $\emptyset'$-change never occurs for these attempts, we trace $U^{\emptyset'}$ into F^A. But then, by the LR-incompleteness of A, we are assured that measure $1 - \mu(U^{\emptyset'})$ worth of attempts will succeed, providing enough restraint to satisfy the requirement.

We can think of this technique in the following way. We are given an approximation $A[s]$ of A such that $A = \lim_s A[s]$. Each time we take a string ρ from $U^{\emptyset'}$ and put it into $F^A[s]$ with some use $\tau = A[s] \upharpoonright u$, we are requesting that $A \supset \tau$ in the limit; the measure of ρ is the strength of the request. If in fact $\tau \subset A$, then the request is successful. Since we will threaten to make $U^{\emptyset'} \subseteq F^A$, the LR-incompleteness of A guarantees that enough requests will be successful to ensure that $\mu(F^A) = 1$. Whether a request is successful or not depends only on whether $\tau \subset A$ in the limit. It does not matter (as far as this basic strategy is concerned) whether A is approximated in a c.e. or Δ^0_2 way. Thus the technique can be used with both c.e. and Δ^0_2 sets A.

4. A c.e. LR-degree incomparable with a given intermediate Δ^0_2 LR-degree

Barmpalias, Lewis and Stephan [4] use an oracle construction to construct a Δ^0_2 set B that is LR-incomparable with a given Δ^0_2 set A of intermediate LR-degree. We strengthen this result to make B c.e., using a full approximation construction.

The analogous theorem for the Turing degrees, namely that for every Δ^0_2 set A of intermediate Turing degree there is (uniformly in A) a c.e. set B Turing incomparable with A, was proved by Sacks [18] using a coding strategy combined with Sacks restraints. A presentation may be found in [17].

Theorem 2. *Let A be a Δ^0_2 set such that $\emptyset <_{LR} A <_{LR} \emptyset'$. There is (uniformly in A) a c.e. set B such that $A \,|_{LR}\, B$.*

Let A be a Δ^0_2 set such that $\emptyset <_{LR} A <_{LR} \emptyset'$, given as a computable approximation $A[s]$ such that

$$\lim_s A(x)[s] = A(x) \quad \text{for all } x.$$

Let $\langle V_e, p_e \rangle$ be a listing of all LR-operators, that is, pairs $\langle V, p \rangle$ of an oracle Σ^0_1 class V and a dyadic rational $p \in (0,1)$ such that

$$\mu(V^X) \leq p \text{ for all } X \in 2^\omega.$$

Let U be a fixed member of a universal oracle Martin-Löf test. We will construct the required c.e. set B, as well as an oracle Σ^0_1 class T, to satisfy the requirements

$$P_e: \qquad T^B \not\subseteq V^A_e$$
$$S_e: \qquad U^A \not\subseteq V^B_e$$

for all $e \in \mathbb{N}$. In fact, we will uniformly build a sequence $T_{\alpha,i}$ of oracle Σ^0_1 classes, where $i \in \mathbb{N}$ and α ranges over nodes of the tree of strategies defined later. We can set $T^B = \cup_{\alpha,i} T^B_{\alpha,i}$ which is a B-Σ^0_1 class. We will ensure that

$$\sum_{\alpha,i} \mu(T^B_{\alpha,i}) < 2^{-1},$$

so by Theorem 1, the requirements P_e ensure that $B \not\leq_{LR} A$. We will use a strategy based on the discussion of section 3 to meet the P requirements; for the S requirements we will use a variation of Sacks restraints adapted for LR degrees, first used by Barmpalias, Lewis and Soskova [3].

4.1. **Outline of the S-strategy.** We use Sacks restraints, adapted to LR-reductions. Sacks restraints were first used by Sacks [18] in the context of c.e. Turing degrees (see [19] for the standard presentation). The technique was adapted to LR-degrees by Barmpalias, Lewis and Soskova [3].

In the Turing case, suppose we are building B and want to ensure that $\Phi^B \neq A$ for a Turing functional Φ and a noncomputable c.e. or Δ^0_2 set A. We can monitor the length of agreement of $\Phi^B = A$, and whenever we see a new computation $\Phi^B(n) = A(n)$ converge, we restrain B on the use u of $\Phi^B(n)$ to preserve that part of the computation. If the restraint is respected and B does not change below u after the restraint is imposed, then we can argue that $\Phi^B \neq A$. If $\Phi^B = A$, then we would be able to compute $A(n)$ by finding a stage in the construction when the length of agreement is above n; at this stage the Φ^B-side of the computation will never change so the approximation to $A(n)$ must be correct. Thus A would be computable, which is a contradiction.

In the case of LR-reductions, we have a fixed member U^A of a universal Martin-Löf test relative to A, and a bounded B-Σ^0_1 class V^B. We want to ensure that $U^A \not\subseteq V^B$. Suppose at some stage we see a string $\sigma \in U^A$ and $\sigma \subseteq V^B$ with use u. We can restrain B up to u in order to preserve the computation $\sigma \subseteq V^B$. Assuming that the restraint is respected, we can also enumerate the string σ into a Σ^0_1 class G. If $U^A \subseteq V^B$, then we will eventually do this for every string $\sigma \in U^A$. Thus $U^A \subseteq G$. But since the B-restraint is respected, each string in G is also in V^B, so $G \subseteq V^B$. Since V^B has measure < 1, so does G. But then U^A is contained in a Σ^0_1 class of bounded measure, which would mean that A is low-for-random, a contradiction. So eventually there must be some string $\sigma \in U^A$ but $\sigma \not\subseteq V^B$, and we succeed in diagonalising against V.

4.2. **Outline of the P-strategy.** Formal details of the P-strategy are given later. We omit the subscript e in the following discussion. We will diagonalise against the LR-operator $\langle V, p \rangle$ by putting a clopen set σ into T^B, waiting for $\sigma \subseteq V^A[s]$, and then removing σ from T^B by enumerating into B if we see $\sigma \subseteq V^A[s]$. If σ is never $\subseteq V^A[s]$ then P is satisfied and the requirement contributes at most $\mu(\sigma)$ to $\mu(T^B)$. If eventually $\sigma \subseteq V^A[s]$ and A does not later change below the use of the computation $\sigma \subseteq V^A[s]$, then $\mu(V^A)$ increases permanently by $\mu(\sigma)$

but $\mu(T^B)$ does not increase. With suitable choice of $\mu(\sigma)$, requirement P must be satisfied after finitely many repetitions of this strategy as $\mu(V^A)$ cannot increase above p.

If however A does later change below the use of $\sigma \subseteq V^A[s]$, then σ may no longer be $\subseteq V^A$ and the attack σ is unsuccessful. We can use the method described in section 3 using the LR-incompleteness of A to impose restraints on A and guarantee that sufficiently many attacks will be successful to satisfy P. Since $A \not\geq_{LR} \emptyset'$, if $U^{\emptyset'} \subseteq F^A$ for some A-c.e. class F^A and member U of a universal oracle Martin-Löf test, then $\mu(F^A) = 1$. When we want to schedule an attack at stage s, we take a string $\rho \in U^{\emptyset'}[s]$ and choose σ with $\mu(\sigma) = \mu(\rho)$. When the attack σ appears successful because $\sigma \subseteq V^A[t]$, we would like to restrain A on the use u of σ. We put ρ into an A-c.e. class F^A with the same use u, then we wait for a $\emptyset'$-change to remove ρ from $U^{\emptyset'}$. If this $\emptyset'$-change never occurs, then $\rho \in U^{\emptyset'}$ permanently and the attack σ is considered unsuccessful; however this can happen for at most $\mu(U^{\emptyset'})$ worth of attacks. Otherwise, $\emptyset'$ eventually changes and ρ is removed from $U^{\emptyset'}$. Then we can remove σ from T^B via a B-change. If later an A-change removes σ from V^A, then the attack σ is unsuccessful. However, since we threaten to trace $U^{\emptyset'}$ into F^A, we are guaranteed that enough ρ's will be permanently in F^A, and hence σ's permanently in V^A, to ensure that $\mu(F^A) = 1$. Since each $\rho \in F^A$ corresponds to an attack σ of the same measure in V^A, we can argue that the P requirement must eventually be satisfied as $\mu(V^A) = 1$ is impossible.

Each attack σ is tied to a computation $\rho \in U^{\emptyset'}[s]$, and the outcome of the attack σ depends on the outcome of the computation $\rho \in U^{\emptyset'}[s]$. If the computation $\rho \in U^{\emptyset'}[s]$ is not permanent, then the attack σ will be removed permanently from T^B and will either succeed or fail, depending on whether $\rho \in F^A$ (and $\sigma \subseteq V^A$). If the computation $\rho \in U^{\emptyset'}[s]$ is permanent, then σ will be permanently in T^B. The attack σ may end up permanently pending (if $\sigma \subseteq V^A$), or permanently waiting (if σ is never $\subseteq V^A$). However we also might have $\sigma \subseteq V^A[s]$ at infinitely many s but $\sigma \not\subseteq V^A$ in the limit; in this case, σ may rotate infinitely often between waiting and pending. In this case σ is permanently in T^B but $\sigma \not\subseteq V^A$ so the P-requirement is satisfied, and σ itself does not cause any enumerations into B. However, during the stages when σ is pending, we will schedule other attacks for P. Each time σ changes from pending to waiting, any attacks scheduled after σ must be removed from T^B. So although σ itself will not cause any B-enumerations, attacks scheduled during σ's pending periods might cause infinitely many B-enumerations, which could conflict with the B-restraints of weaker-priority S-requirements. This conflict is resolved by having the P-strategy play an infinitary outcome each time σ moves between pending and waiting; the S-strategies below the infinitary outcome will only believe a computation if its use is below that of any attacks scheduled after σ.

We will argue that if P is not satisfied then $U^{\emptyset'} \subseteq F^A$, F^A has measure 1, and so V^A must have measure 1 also since every string in F^A corresponds to a successful attack in V^A. For this argument to work, we require that successful attacks are disjoint. This is slightly complicated by the Δ^0_2 approximation of A. At a stage s_0 we may have an attack σ which is succeeding, ie $\sigma \subseteq V^A[s_0]$ and its corresponding string $\rho \in F^A[s_0]$. At $s_1 > s_0$ the attack σ might be failing ($\sigma \not\subseteq V^A[s_1]$ and $\rho \notin F^A[s_1]$) because the approximation to A has changed. We might now schedule a new attack σ', which might overlap (as a clopen set) with σ. However, later on at $s_2 > s_1$ the approximation to A might change back to its state at s_0, and σ would become succeeding again. If we keep working with σ', we risk having two non-disjoint attacks. To account for this, when we create a new attack σ' we record the state of all earlier attacks, in the form of a suitable initial segment γ of $A[s]$. We choose $|\gamma|$ longer than the use of any computations relevant to earlier attacks σ which might later become succeeding if A reverts back to an earlier approximation. We will only work with σ' at stages when $\gamma \subset A[s]$. We call γ the *state* of the attack σ'. Note that this is not necessary if A is in fact c.e., since a c.e. approximation cannot revert to a previous state.

Since we want to illustrate the technique outlined in section 3 in generality, the form of the P-strategy we are using here is slightly more general than is necessary for this specific construction. To meet the P-requirements of this theorem we could trace strings from $U^{\emptyset'}$ directly into T^B, rather than via an intermediate set F^A. We could thus slightly simplify the notation by eliminating the class F^A. However the notion of state and the infinitary outcomes would still be necessary. In order to establish notation that is suitable for more general applications of this technique, we will not make this simplification in this construction.

4.3. **The priority tree.** The construction takes place on an $\omega+1$ branching tree, with nodes labelled either $P_{e,i}$ or S_e for some $e, i \in \mathbb{N}$. Nodes are labelled according to their length: if $|\alpha| = 2e$ then α is labelled S_e and has a single outcome 0 (that is, there is a single child node $\alpha^\frown 0$ extending α on the tree). If $|\alpha| = 2\langle e, i\rangle + 1$ then α is labelled $P_{e,i}$ and has $\omega+1$ outcomes (children)

$$\infty(0) <_L \infty(1) <_L \infty(2) <_L \ldots <_L f.$$

The ordering of outcomes $<_L$ induces an ordering on the tree; for nodes α, β, $\alpha <_L \beta$ indicates that α is to the left of β, and $\alpha < \beta$ indicates that $\alpha <_L \beta$ or $\alpha \subset \beta$. We refer to nodes labelled $P_{e,i}$ for some e, i as *P-nodes*, or as P_e-nodes if the index e is significant, and similarly for S-nodes.

During the construction we will define *approximations to the true path* TP_s, which are the nodes of the tree which are active at stage s. Say that s is an *α-stage*, or alternatively that α is *accessible* at s, if $\alpha \subseteq \mathrm{TP}_s$.

4.4. ***P*-requirements.** Each P-node pursues an independent copy of the P-strategy. Fix a computable listing $\alpha_0, \alpha_1, \ldots$ of all P-nodes on the tree. Recall that p_e is an upper bound on the measure $\mu(V_e^X)$ of the e'th LR-operator. If α is a P_e-node (for some $e \in \mathbb{N}$), let n be its position in the above ordering. Let m_α be the least number m such that $2^{-m} < 2^{-n-2} \cdot p_e$. Each P-node α has a counter $c(\alpha, s)$ which is the number of times that node α has been reset by the end of stage s. At any stage, α works with oracle Σ^0_1-classes T_α and F_α, and a member $U_\alpha^{\emptyset'}$ of a universal Martin-Löf test relative to the halting problem. Actually, α works with a sequence of oracle Σ^0_1-classes $T_{\alpha,i}$ and Martin-Löf test members $U^{\emptyset'}_{\alpha,i}$, $i \in \mathbb{N}$. Each time α is reset it empties F_α, abandons the previous $T_{\alpha,i}, U^{\emptyset'}_{\alpha,i}$ and starts working with $T_{\alpha,i+1}, U^{\emptyset'}_{\alpha,i+1}$ instead. To be precise, at stage s, α will work with $T_{\alpha,c(\alpha,s)}$ and $U^{\emptyset'}_{\alpha,c(\alpha,s)}$, where $U_{\alpha,i}$ is the $m_\alpha + i + 1$'th member of the universal oracle Martin-Löf test. For brevity we write T_α and U_α to refer to the appropriate class $T_{\alpha,i}, U_{\alpha,i}$ which is in use at the time; T_α and U_α may be considered pointers to $T_{\alpha,c(\alpha,s)}$ and $U_{\alpha,c(\alpha,s)}$ at each stage s. If α is reset only finitely often, then T_α, U_α are eventually fixed. We will ensure that $\mu(T^X_{\alpha,i}[s]) \leq 2^{-m_\alpha - i - 1}$ for all $X \in 2^\omega$ and all s, so that, setting $T = \bigcup_\alpha \bigcup_i T_{\alpha,i}$, we have

$$\mu(T^B) \leq \sum_\alpha \sum_i 2^{-m_\alpha - i - 1} \leq \sum_\alpha 2^{-m_\alpha} \leq \frac{1}{2}, \tag{5}$$

where α ranges over all P-nodes.

Let α be a P_e-node. The node α will attempt to meet its requirement by putting certain clopen sets of reals (*attacks*) σ into T^B_α, waiting until $T^B_\alpha \subseteq V^A_\alpha$, then removing the clopen set σ from T^B_α and attempting to restrain A to keep $\sigma \subseteq V^A_\alpha$. Each time, α causes the measure of V^A_α to increase by $\mu(\sigma)$, while $\mu(T^B_\alpha)$ does not increase. Since $\mu(V^A_\alpha)$ is bounded by $p_\alpha < 1$, this can only happen finitely often (with a suitable choice of $\mu(\sigma)$) before some attack satisfies the requirement because $\sigma \not\subseteq V^A_\alpha$.

An *attack* σ is a finite prefix-free set of strings (representing a clopen set of reals) which we treat as a single unit. It is possible that two attacks may be created at different times in the construction which have the same set of strings, but in this case we consider them to be distinct attacks (formally, we may consider an attack as the finite set of strings along with the stage at which it is scheduled, though we will not do this explicitly). When we put an attack into $T^B_\alpha[s]$ we put each string from σ into $T^B_\alpha[s]$ with the same use; we write $\sigma \in T^B_\alpha[s]$ to mean that each string in σ is in $T^B_\alpha[s]$ with the same use. At most one attack is scheduled (created) at each stage. If an attack σ is scheduled by node α then we say that σ is an α-attack. The lifecycle of an attack is as follows. It is *scheduled* at some stage t. At future stages it is either *current* or not current; if it is current then it is either *waiting* for V^A_α, *pending* $\emptyset'$-permission, *succeeding* or *failing*. These terms are defined later.

We consider each $U_{\alpha,i}$ as a c.e. set of axioms. An axiom is a pair $\langle \rho, \tau\rangle$ asserting that $\rho \in U^X$ if $\tau \subset X$. Since we are working with $U^{\emptyset'}$ as we approximate $\emptyset'$, we are only interested in those axioms such that $\tau \subseteq \emptyset'[s]$ at some stage. An axiom $\langle \rho, \tau\rangle$ is *valid* at stage s if the tuple $\langle \rho, \tau\rangle$ has been enumerated into U by stage s and $\tau \subset \emptyset'[s]$. Fix α, i and let

$$\langle \rho_0, \tau_0\rangle, \langle \rho_1, \tau_1\rangle \dots \tag{6}$$

be a list of the computations (axioms) from $U_{\alpha,i}$ which are valid at some stage, ordered first by the least stage at which they are valid and then by the usual length/lexicographical ordering on the ρ_j. When an α-attack is created, it is associated with one of these computations. Suppose that attack σ is associated with axiom $\langle \rho_k, \tau_k\rangle$. We write $\operatorname{rank}(\sigma)$ to denote the position k of the axiom in the above list, $\rho(\sigma)$ to denote the string ρ_k, $\hat\rho(\sigma)$ to denote the computation $\langle \rho_k, \tau_k\rangle$, and $u(\sigma)$ to denote $|\tau_k|$.

Attack σ also has a *state* $\gamma(\sigma)$, defined when σ is scheduled, which is an initial segment of A at the stage when σ is scheduled. The attack σ is *current* at a later stage $s+1$ if $\gamma(\sigma) \subset A[s]$; at any stage, we only work with attacks which are current.

When we wish to restrain a computation $\sigma \subseteq V^A_\alpha[s]$, we will put the string $\rho(\sigma)$ into $F^A_\alpha[s+1]$ by defining a new computation with use v larger than that of the computation $\sigma \subseteq V^A_\alpha[s]$. We say that the axiom $\langle \rho, A[s] \upharpoonright v\rangle$ is in F_α *on account of* σ. Every axiom in F_α is on account of some attack. Two distinct axioms $\langle \rho, \tau\rangle, \langle \rho', \tau'\rangle \in F_\alpha$ may be on account of different attacks even if $\rho = \rho'$ (although then $\tau \neq \tau'$). We say that $\rho \in F^A_\alpha[s]$ on account of σ if $\rho \in F^A_\alpha[s]$ due to an axiom $\langle \rho, \tau\rangle \in F_\alpha[s]$ on account of σ.

Schedule an α-attack at stage $t+1$ by taking the least k (if it exists) such that $\langle \rho_k, \tau_k\rangle$ is valid at $t+1$ and $\rho_k \notin F^A_\alpha[t]$, and choosing the least clopen set $\sigma \subseteq 2^\omega - V^A_\alpha[t]$ with $\mu(\sigma) = \mu(\rho_k)$. Set $\rho(\sigma) = \rho_k$, $\hat\rho(\sigma) = \langle \rho_k, \tau_k\rangle$, $u(\sigma) = |\tau_k|$ and $\operatorname{rank}(\sigma) = k$. Let w be the maximum use of any computation $\langle \rho', \tau'\rangle \in F_\alpha[t]$ on account of any attack σ' with $\operatorname{rank}(\sigma') < k$, and define the state $\gamma(\sigma)$ to be $A[t] \upharpoonright w$. Put σ into $T^B_\alpha[t+1]$ with fresh use. Note that, when scheduling an attack, a suitable choice for σ will always exist since

$$\mu(\rho_k) \le \mu(U^{\emptyset'}_\alpha[t]) \le 1 - p_\alpha \le \mu(2^\omega - V^A_\alpha[t])$$

by choice of m_α. It is possible that sometimes k will not exist (if $U^{\emptyset'}_\alpha[t] \subseteq F^A_\alpha[t]$). In this case, do nothing; no attack is scheduled at $t+1$.

At certain stages $s+1$ we will *implement* σ by putting $\rho(\sigma)$ into $F^A_\alpha[s+1]$ with some use v; that is, enumerating a new axiom $\langle \rho(\sigma), A[s] \upharpoonright v\rangle$ into $F_\alpha[s+1]$. We declare that the new computation is on account of σ.

The attack σ which was scheduled at $t+1$ is *failing* at $s+1 > t+1$ if it is current at $s+1$, $\emptyset'[s] \upharpoonright u(\sigma) \neq \emptyset'[t] \upharpoonright u(\sigma)$ and $\rho(\sigma)$ is not in $F^A_\alpha[s]$ on account of σ. σ is *succeeding* at $s+1$ if it is current at $s+1$, $\emptyset'[s] \upharpoonright u(\sigma) \neq \emptyset'[t] \upharpoonright u(\sigma)$ and $\rho(\sigma)$ is in $F^A_\alpha[s]$ on account of σ.

Attack σ is *waiting* at stage $s+1$ if it is current at $s+1$, $\emptyset'[s] \upharpoonright u(\sigma) = \emptyset'[t] \upharpoonright u(\sigma)$, but $\rho(\sigma)$ is not in $F^A_\alpha[s]$ on account of σ. This is the case when $\sigma \not\subseteq V^A_\alpha[s]$. σ is *pending* at stage $s+1$ if it is current at $s+1$, $\emptyset'[s] \upharpoonright u(\sigma) = \emptyset'[t] \upharpoonright u(\sigma)$ and $\rho(\sigma)$ is in $F^A_\alpha[s]$ on account of σ. In this case we are waiting for a $\emptyset'$-change before we remove σ from T^B_α.

It is possible that for some σ we may have $\sigma \subseteq V^A_\alpha[s]$ for infinitely many s but $\sigma \not\subseteq V_\alpha$ in the limit. Such a σ may be implemented infinitely often, with $A[s]$ always changing below the use of the new F_α computation. In this case, $\sigma \in T_\alpha^B$ permanently but $\sigma \not\subseteq V_\alpha^A$ so the requirement P_α is satisfied. Although σ itself will not cause infinitely many enumerations into B, attacks of rank $>$ rank(σ) may cause infinitely many B-enumerations since any such attack will not be permanently current and will eventually need to be removed from T_α^B. To allow the lower-priority negative requirements to work with these potentially infinitary enumerations, each time attack σ is implemented we access an infinitary outcome $\alpha^\frown\infty(k)$ for $k = \text{rank}(\sigma)$. The negative requirements below this outcome will only believe a computation if the use of the computation is less than that of any α-attack of rank $> k$ which is in $T_\alpha^B[s]$.

Let β be an S-node below some infinitary outcome $\alpha^\frown\infty(n)$. The node β will only believe a computation if the use of the computation is less than the use of any α-attack σ which is currently in $T_\alpha^B[s]$ and has rank$(\sigma) > n$. Precisely, suppose that β is an S-node and $\tau \subseteq V^B[s]$ with use u for some clopen set τ and oracle Σ_1^0-class V. The computation $\tau \subseteq V^B[s]$ is *β-believable* if there does not exist a P-node α such that

- $\alpha^\frown\infty(n) \subseteq \beta$ for some n, and
- there is an α-attack σ with rank$(\sigma) > n$ and $\sigma \in T_\alpha^B[s]$ with use $\leq u$.

If in fact $\tau \subseteq V^B$ and β is on the true path, then eventually the computation will be β-believable.

We observe that the construction can be considerably simplified in the case that the set A is low ($A' \equiv_T \emptyset'$). In this case we can avoid the infinitary outcomes by a technique similar to Robinson guessing (see [19] §XI.3). We can take a computable function $h : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$ such that

$$\lim_s h(\sigma, e, s) = \begin{cases} 1 & \text{if } \sigma \subseteq V_e^A \\ 0 & \text{otherwise} \end{cases}$$

and believe a computation $\sigma \subseteq V_e^A[s]$ only if $h(\sigma, e, s) = 1$. In this case, if $\sigma \subseteq V_e^A[s]$ at infinitely many s but not in the limit, then σ will only be implemented finitely often before $h(\sigma, e, s)$ reaches its limiting value of 0. Hence the infinitary outcomes are not necessary, and we could use the strategy in a finite injury construction.

4.5. ***S*-requirements.** Let α be an S-node. Define the *length of agreement*

$$l(\alpha, s) = \max\Big\{i : \langle \rho_i, \tau_i \rangle \in U[s] \text{ and } \forall j \leq i(\rho_i \subseteq V_\alpha^B[s] \text{ by an } \alpha\text{-believable computation, or } \rho_i \notin U^A[s])\Big\}.$$

Since A is Δ_2^0, we also need the *modified length of agreement*

$$m(\alpha, s) = \max\Big\{i : \exists \alpha\text{-stage } t \leq s\Big(i \leq l(\alpha, t) \ \wedge \ B[s] \restriction u = B[t] \restriction u\Big)\Big\}$$

where u is the maximum use of all computations $\rho_j \subseteq V_\alpha^B[t]$ for $j \leq i$ with $\rho_j \in U^A[t]$.

Each S-node has a restraint r_α which is initially 0 and is set explicitly during the construction. Let $R(\alpha, s) = \max\{r_\beta[s] : \beta < \alpha\}$ be the total restraint imposed by nodes of higher priority than α. To *reset* an S-node α at stage s means to set $r_\alpha[s] = 0$.

4.6. **Some conventions.** We assume that when a string is put into any $T_{\alpha,i}^B$ with some use u, it remains there until the number u is explicitly enumerated into B. In particular, the string remains in $T_{\alpha,i}^B$ even if numbers $< u$ enter B.

We use the 'hat-trick' for the enumeration of U^A. Let $a_0 = 0$ and for $s > 0$ let a_s be the least number such that $A(a_s)[s] \neq A(a_s)[s-1]$, or $a_s = s$ if such number does not exist. Let

$$\widehat{U^A}[s] := U^{A\restriction a_s}[s] = \{\sigma : \sigma \text{ is in } U^A[s] \text{ with use} \leq a_s\}.$$

Henceforth we omit the hat and write $U^A[s]$ to mean $\widehat{U^A}[s]$. The hat-trick ensures that $\sigma \in U^A[s]$ for all but finitely many s iff $\sigma \in U^A$.

4.7. **The construction.** Initially, B and all classes $T_{\alpha,i}, F_\alpha$ are empty, and r_α is zero for all S-nodes α.

At stage 0, do nothing.

At stage $s+1$, define TP_{s+1} inductively as below. After TP_{s+1} is defined, reset all nodes $\beta >_L \mathrm{TP}_{s+1}$.

Suppose that $\mathrm{TP}_{s+1} \upharpoonright n$ is defined, for $n \geq 0$. If $n = s+1$ then stop defining TP_{s+1}. Otherwise let $\alpha = \mathrm{TP}_{s+1} \upharpoonright n$ and go to the appropriate case below.

• α **is a** P**-node**. Let s' be the previous stage when α was accessible, or 0 if never. Check if any of the following hold:

(I) there is an α-attack σ such that σ is in $T^B_\alpha[s]$ but σ is not current at $s+1$;
(II) there is a current α-attack σ that is failing or succeeding at $s+1$ and $\sigma \in T^B_\alpha[s]$;
(III) there is a current α-attack σ that is pending or waiting at $s+1$ but $\sigma \notin T^B_\alpha[s]$;
(IV) there is a current α-attack σ which is waiting at $s+1$ and $\sigma \subseteq V^A_\alpha[s]$ (as sets of reals);
(V) no current α-attacks are waiting at $s+1$.

Go to the least case below which holds.

(I) or (II) hold. For each such σ, remove σ from $T^B_\alpha[s+1]$ by enumerating the use of the computation $\sigma \in T^B_\alpha[s]$ into $B[s+1]$. Let k be the minimum rank of all such σ, and reset $\alpha^\frown\infty(k)$ and all nodes of lower priority. Stop defining TP_{s+1}.

(III) holds. For each such σ, add σ to $T^B_\alpha[s+1]$ with fresh use. Stop defining TP_{s+1}.

(IV) holds. For the least such σ, let v be the maximum of $|\gamma(\sigma)|$ and the use of the computation $\sigma \subseteq V^A_\alpha[s]$, and implement σ by defining $\rho(\sigma) \in F^A_\alpha[s+1]$ with use v. Let $k = \mathrm{rank}(\sigma)$ and let $\mathrm{TP}_{s+1} \upharpoonright n+1 = \alpha^\frown\infty(k)$.

(V) holds. Schedule a new α-attack at $s+1$. Stop defining TP_{s+1}.

None of (I)-(V) hold. Let $\mathrm{TP}_{s+1} \upharpoonright n+1 = \alpha^\frown f$.

• α **is an** S**-node**. Let r be the maximum use of all computations $\rho_j \subseteq V^B_\alpha[s]$ for $j \leq m(\alpha, s)$ with $\rho_j \in U^A[s]$. If $r > r_\alpha[s]$ then set $r_\alpha[s+1] = r$, reset all nodes of lower priority than α and stop defining TP_{s+1}. Otherwise let $\mathrm{TP}_{s+1} = \alpha^\frown 0$.

End of construction.

4.8. **Verification.** First we deal with the P-requirements. We give some lemmas to clarify the relations between attacks.

Lemma 3. Let α be a P-node. If σ is an α-attack such that $\mathrm{rank}(\sigma) = k$ and σ is current at stage $s+1$, then for every $j < k$ such that the j'th computation from the list (6) is valid at $s+1$ there is an α-attack σ' with $\rho(\sigma') = \rho_j$ and $\mathrm{rank}(\sigma') \leq j$ which is current at $s+1$.

Proof. Suppose that there is a $j < k$ such that the j'th computation $\langle \rho_j, \tau_j \rangle$ is valid at $s+1$ but there is no current attack with rank j. Let $t+1 < s+1$ be the stage when σ was scheduled. Since $\mathrm{rank}(\sigma) > j$, the computation $\langle \rho_j, \tau_j \rangle$ is valid at $t+1$, and we must have $\rho_j \in F^A_\alpha[t]$. But then there must be a current α-attack σ' pending at $t+1$ with $\rho(\sigma') = \rho_j$ and $\mathrm{rank}(\sigma') \leq j$. But $\gamma(\sigma') \subseteq \gamma(\sigma)$, and σ is current at $s+1$, so σ' must also be current at $s+1$. □

The next lemma states that no two attacks associated with the same computation from $U^{\emptyset'}_\alpha$ are simultaneously current, and that any two attacks which both have strings in $F^A_\alpha[s]$ are disjoint.

Lemma 4. For any P-node α and any stage s, if distinct α-attacks σ, σ' are both current at s then $\mathrm{rank}(\sigma) \neq \mathrm{rank}(\sigma')$. If $\rho(\sigma), \rho(\sigma')$ are both in $F^A_\alpha[s]$ on account of σ, σ' respectively, then $\rho(\sigma) \neq \rho(\sigma')$ and $\sigma \cap \sigma' = \emptyset$ (as sets of reals).

Proof. Let σ, σ' be as in the claim. Assume for contradiction that $\mathrm{rank}(\sigma) = \mathrm{rank}(\sigma')$, and let $t+1, t'+1$ be the stages when σ, σ' respectively are scheduled. Suppose w.l.o.g. that $t < t'$. Then $\gamma(\sigma) \subseteq \gamma(\sigma') \subset A[t']$, so σ must be current at $t'+1$, and is either waiting, pending, failing or succeeding. σ cannot be waiting at $t'+1$ or the attack σ' would not be scheduled. Nor can σ be failing or suceeding, as the computation $\hat{\rho}(\sigma)$ $(= \hat{\rho}(\sigma'))$ is valid at $t'+1$. Finally if σ is pending at $t'+1$ then $\rho(\sigma) \in F^A_\alpha[t']$, and σ' would be scheduled with $\rho(\sigma') \neq \rho(\sigma)$. So $\mathrm{rank}(\sigma) \neq \mathrm{rank}(\sigma')$.

Suppose that $\rho(\sigma), \rho(\sigma')$ are in $F^A_\alpha[s]$ on account of σ, σ'. Then both σ, σ' are current at $s+1$ by choice of the use of the F_α computations, so $\mathrm{rank}(\sigma) \neq \mathrm{rank}(\sigma')$ by the above. Suppose w.l.o.g. that $\mathrm{rank}(\sigma) < \mathrm{rank}(\sigma')$ and let $t+1, t'+1$ be the stages when σ, σ' are scheduled. By Lemma 3, $t < t'$. Then σ must be pending, failing or succeeding at $t'+1$. If σ is pending or succeeding at $t'+1$, then $\sigma \subseteq V^A_\alpha[t']$ and σ' will be chosen disjoint from σ. If σ is failing at $t'+1$, then we cannot have $\rho(\sigma) \in F^A_\alpha[s]$ on account of σ since $A[t']$ and $A[s]$ agree on the use of any such computation. So σ and σ' are disjoint. □

The next lemma verifies that $\mu(T^B) < 1$.

Lemma 5. For any P-node α and $s \in \mathbb{N}$,

$$\mu(T^B_\alpha[s]) \leq 2^{-m_\alpha - c(\alpha,s)-1}.$$

Proof. Suppose that $T^B_\alpha[s]$ is not empty, and let $t+1$ be the greatest stage $\leq s$ when an attack is added to $T^B_\alpha[t+1]$. At stage $t+1$, (I) and (II) do not hold for any α-attack, as otherwise nothing would be added to $T^B_\alpha[t+1]$. So for every attack $\sigma \in T^B_\alpha[t+1]$, σ is pending or waiting at $t+1$ and $\rho(\sigma) \in U^{\emptyset'}_\alpha[t]$. Since $\mu(\sigma) = \mu(\rho(\sigma))$ and $T^B_\alpha[s] \subseteq T^B_\alpha[t+1]$, we have $\mu(T^B_\alpha[s]) \leq \mu(U^{\emptyset'}_\alpha[t]) \leq 2^{-m_\alpha - c(\alpha,s)-1}$. □

Note that since A is Δ^0_2 and the string $\gamma(\sigma)$ is finite, each attack σ is eventually either permanently current (that is, σ is current at all $s >$ some s_0) or permanently not current, depending on whether $\gamma(\sigma) \subset A$.

The following lemma describes the fate of α-attacks for α on the true path.

Lemma 6. Suppose that α is a P-node which is accessible infinitely often and is reset only finitely often. Let s_0 be the last stage at which α is reset (or 0 if never). Suppose that σ is an α-attack scheduled at $t+1 > s_0$ such that σ is eventually permanently current and no α-attack σ' with $\mathrm{rank}(\sigma') < \mathrm{rank}(\sigma)$ is implemented infinitely often. Then at least one of the following holds:

(A) σ is implemented infinitely often;
(B) σ is permanently pending after some s (that is, σ is pending at every $s' > s$);
(C) σ is permanently succeeding after some s;
(D) σ is permanently failing after some s;
(E) $\sigma \not\subseteq V^A_\alpha[s']$ at every α-stage s' after some stage s.

Proof. We use induction on $\mathrm{rank}(\sigma)$. Let σ be as in the claim and assume inductively that there is a stage s_1 such that σ is permanently current after s_1 and all permanently current α-attacks of rank $< \mathrm{rank}(\sigma)$ scheduled after s_0 satisfy one of (B)-(E) for $s = s_1$.

First consider the case that the computation $\hat{\rho}(\sigma)$ is not permanent. Then σ is implemented only finitely often (possibly never), with finitely many computations $\langle \rho(\sigma), \tau_0 \rangle, \ldots \langle \rho(\sigma), \tau_n \rangle$ in F_α on account of σ. If $A \supset \tau_i$ for some i then eventually σ is permanently succeeding and (C) holds; otherwise σ is permanently failing and (D) holds.

Next consider the case that the computation $\hat{\rho}(\sigma)$ is permanent. If σ is implemented infinitely often then (A) holds. Suppose that σ is implemented only finitely often and (E) does not hold; that is, $\sigma \subseteq V^A_\alpha[s']$ for infinitely many α-stages s'. Each time this occurs after s_1, σ will be implemented unless some existing computation $\langle \rho(\sigma), \tau \rangle \in F_\alpha[s']$ on account of σ is valid. Since σ is implemented only finitely often, and as A is Δ^0_2, eventually the approximation $A[s]$ will settle on the use of these computations. After this point, one of the computations must be permanently valid, and $\rho(\sigma) \in F^A_\alpha$ permanently on account of σ. Thus (B) holds. □

In fact, (A)-(E) are mutually exclusive except for the pair (D) and (E), though this is not needed in the verification.

As usual, the true path consists of the leftmost infinitely-often visited nodes. Since the tree is infinitely branching, and since we sometimes stop defining TP_s early, we must verify that the true path exists. This lemma verifies that P-nodes do not cause the true path to be finite; Lemma 10 does the same for S-nodes. We simultaneously verify that a P-node on the true path satisfies its P-requirement.

Lemma 7. Suppose that α is a P-node that is reset only finitely often and is accessible infinitely often. Then $T^B_\alpha \not\subseteq V^A_\alpha$. Furthermore, either some outcome $\alpha^\frown\infty(i)$ is accessible infinitely often, or after some stage s_0 outcome $\alpha^\frown f$ is accessible at every α-stage.

Proof. Suppose first that some α-attack is implemented infinitely often; let σ be the least α-attack by rank which is implemented infinitely often. Since σ is permanently current, and the computation $\hat\rho(\sigma)$ is permanently valid, σ is permanently in T^B_α. However, $\sigma \not\subseteq V^A_\alpha$, as otherwise σ would be implemented only finitely many times before some computation $\rho(\sigma) \in F^A_\alpha$ on account of σ was permanent. So $T^B_\alpha \not\subseteq V^A_\alpha$ and P_e is satisfied. Furthermore, every time σ is implemented, outcome $\infty(k)$ is accessed, where $k = \mathrm{rank}(\sigma)$.

Suppose then that no α-attack is implemented infinitely often. Inductively, by Lemma 6, every permanently current α-attack satisfies one of (B)-(E). We claim that some attack satisfies (E), and hence $T^B_\alpha \not\subseteq V^A_\alpha$, only finitely many α-attacks are scheduled, and $\alpha^\frown f$ is accessible at all but finitely many α-stages. Suppose for contradiction that there is no attack which satisfies (E). Then infinitely many α-attacks are scheduled. We first argue that $U^{\emptyset'}_\alpha \subseteq F^A_\alpha$.

We argue by induction on the U_α computations $\langle \rho_i, \tau_i\rangle$ that for every string $\rho \in U^{\emptyset'}_\alpha$ there is an α-attack σ which is permanently pending or permanently succeeding with $\rho(\sigma) = \rho$. For $\rho_i \in U^{\emptyset'}_\alpha$, assume inductively that there is a stage s_1 such that for every computation $\rho_j \in U^{\emptyset'}_\alpha, j < i$ there is an attack σ_j with $\rho(\sigma_j) = \rho_j$ which is permanently pending or succeeding by s_1. Let s_2 be a stage such that the computation $\rho_i \in U^{\emptyset'}_\alpha[s_2]$ is correct, and any computations $\rho_j \in U^{\emptyset'}_\alpha[s_2]$ for $j < i$ are correct.

By choice of s_2, if an α-attack is scheduled at $s > s_2$ and there is no attack σ pending at s with $\rho(\sigma) = \rho$, then the newly scheduled attack will have rank i. Furthermore, $\gamma(\sigma')$ is fixed for all such attacks scheduled after s_2. Therefore eventually there must be an attack σ which is permanenly current and has $\rho(\sigma) = \rho$; by Lemma 6 and the assumptions it must eventually be permanently pending. This establishes that $U^{\emptyset'}_\alpha \subseteq F^A_\alpha$.

Since $\emptyset' \not\leq_{LR} A$, we must have $\mu(F^A_\alpha) = 1$. By Lemma 4, every string ρ in F^A_α corresponds to an attack of the same measure in V^A_α, and distinct strings correspond to disjoint attacks. Therefore $\mu(V^A_\alpha) = 1 > p_\alpha$, a contradiction.

Some α-attack must therefore satisfy (E) of Lemma 6, and hence $T^B_\alpha \not\subseteq V^A_\alpha$ and P_e is satisfied. Since condition (V) in the construction holds for only finitely many α-stages, only finitely many α-attacks are scheduled. Since each attack satisfies one of (I)-(IV) only finitely often, outcome f is accessed at all but finitely many α-stages. □

Now we deal with the S-requirements. First we verify that an S-node's restraint is respected.

Lemma 8. For an S-node α and stage s, if $r_\alpha[s] \neq 0$ and α is not reset at stage $s+1$ then $B[s+1] \upharpoonright r_\alpha[s] = B[s] \upharpoonright r_\alpha[s]$.

Proof. Suppose that α is not reset at $s+1$, $r = r_\alpha[s] \neq 0$ but $B[s+1] \upharpoonright r \neq B[s] \upharpoonright r$. Some P-node β must enumerate a number $x < r$ into $B[s+1]$ in order to remove an attack σ from $T^B_\beta[s+1]$. We must have $\beta < \alpha$ since all nodes of lower priority than α are reset when r_α is set nonzero, and thereafter any attacks would be put into T with use $> r$. Let σ be the least β-attack by rank which is removed from $T^B_\beta[s+1]$, and let $k = \mathrm{rank}(\sigma)$. Since all nodes of lower priority than $\beta^\frown\infty(k)$ are reset at $s+1$, we must have $\beta^\frown\infty(j) \subseteq \alpha$ for some $j < k$. Let $t+1$ be the greatest stage $< s$ when σ was put into $T^B_\beta[t+1]$, and let $s'+1$ be the greatest stage $< s+1$ when r_α was increased above x. If $t+1 < s'$ then $\sigma \in T^B_\beta[s']$, and since $\text{rank}(\sigma) > j$ and $x < r$ the computation $\rho \subseteq V^B_\alpha[s']$ with use r would not be α-believable. So r_α would not be set nonzero at $s'+1$, contradicting the choice of s'. If $t+1 > s'$ then σ would be put into $T^B_\beta[t+1]$ with fresh use $x > r$, contradicting $x < r$. Also $s' = t$ is impossible since we stop defining TP_{t+1} once β takes action. So $B[s] \upharpoonright r = B[s+1] \upharpoonright r$. □

The next lemma verifies that true computations will eventually be believable, with respect to an S-node on the true path.

Lemma 9. Let α be an S-node such that α is accessible infinitely often and α is reset only finitely often. Suppose that $\tau \subseteq V^B_\alpha$. Then there is a stage s_0 such that the computation $\tau \subseteq V^B_\alpha[s]$ is α-believable at all $s \geq s_0$.

Proof. Suppose that α is as in the claim, and let s_0 be the first α-stage such that $\tau \subseteq V^B_\alpha[s_0]$ with use u, and $B[s_0] \upharpoonright u = B \upharpoonright u$. Then the computation $\tau \subseteq V^B_\alpha$ is believable at s_0. If it were not, then there must be a P-node β with $\beta^\frown\infty(k) \subseteq \alpha$ for some k, and a β-attack σ with $\text{rank}(\sigma) > k$ and $\sigma \in T^B_\beta[s_0]$ with use $< u$. Note that β's computation $\rho_k \in U^{\emptyset'}$ must be permanent, since otherwise $\beta^\frown\infty(k)$, and hence α, would not be accessible infinitely often. Let $t+1$ be the stage when σ was scheduled. By Lemma 3, $\rho_k \in F^A_\beta[t]$ on account of some attack with rank $\leq k$, so $\gamma(\sigma)$ is greater than the use of $\rho_k \in F^A_\beta[t]$. By Lemma 7 there is a stage $s_1 \geq s_0$ when some β-attack σ' with $\rho(\sigma') = \rho_k$ is implemented. At s_1, the computation $\rho_k \in F^A_\beta[t]$ is no longer valid, so σ cannot be current at s_1. Therefore σ must have been removed from T^B_β before s_1 but after s_0 via a B-enumeration, which contradicts the choice of s_0. So the computation $\tau \in V^B_\alpha$ is believable by s_0. □

Now we verify that the restraint $m(\alpha, s)$ reaches a limit for nodes on the true path.

Lemma 10. Let α be an S-node such that α is accessible infinitely often and α is reset only finitely often. Then

$$\lim_s m(\alpha, s) < \infty.$$

Proof. By Lemma 8, α's restraint is respected at all $s \geq$ some s_0. After s_0, $m(\alpha, s)$ does not decrease. Suppose that $\lim_s m(\alpha, s) = \infty$. Enumerate a Σ^0_1-class G as follows: put the string ρ_i into G at α-stage $s \geq s_0$ if $\rho_i \in U^A[s]$ and $m(\alpha, s) \geq i$. Then $U^A \subseteq G$, since eventually $m(\alpha, s) \geq i$ for every $\rho_i \in U^A$. Also, since the restraint $r(\alpha, s)$ is greater than the use of $\rho_i \subseteq V^B_\alpha[s]$ and is respected after s, we have $G \subseteq V^B_\alpha$. So $\mu(G) \leq \mu(V^B_\alpha) < 1$. But this gives $A \leq_{LR} \emptyset$ since G is Σ^0_1, which contradicts $\emptyset <_{LR} A$. Therefore $m(\alpha, s) < i$ for some i and all s. □

Lemma 11. The true path $\text{TP} = \liminf_s \text{TP}_s$ exists and is infinite, and each node on it is reset only finitely often.

Proof. The root node is on TP_s for all s and is never reset. Inductively assume that $\alpha = \liminf_s \text{TP}_s \upharpoonright n$ for $n > 0$, and that α is reset only finitely often. If α is a P-node, then by Lemma 7 there is some outcome (either f or $\infty(k)$ for some k) that is accessible infinitely often. If α is an S-node, then Lemma 10 guarantees that the child $\alpha^\frown f$ is accessible at all but finitely many α-stages, since the definition of TP_s is only ended at α if r_α and $m(\alpha, s)$ increase. Therefore $\beta = \liminf_s \text{TP}_s \upharpoonright n+1$ exists.

Now we verify that β is reset only finitely often. The situations where β might be reset are when (I) or (II) holds for some P-node $\gamma \subset \beta$, when some S-node $\delta \subset \beta$ increases its restraint, or when $\text{TP}_s <_L \beta$. The last can happen only finitely often by induction assumption. By Lemma 10, each S-node $\delta \subset \beta$ increases its restraint only finitely often.

Suppose then that γ is a P-node and $\gamma^\frown f \subseteq \beta$. By Lemma 7 there is an attack σ satisfying (E) at all γ-stages after some s_0. After s_0, no new attacks will be scheduled by γ, and existing attacks can cause β to be reset only finitely often after s_0.

If $\gamma^\frown\infty(k) \subseteq \beta$ for some k, then k is the least such that some γ-attack of rank k is implemented infinitely often. By Lemma 3, there are only finitely many γ-attacks of rank $< k$, and they can cause β to be reset due to (I) or (II) only finitely often. □

Lemma 12. Each requirement P_e is satisfied.

Proof. Let α be the P_e-node on TP. By Lemma 7, there is an α-attack σ that is permanently in T^B_α but $\sigma \not\subseteq V^A_\alpha$. (In the case that $\alpha^\frown\infty(k) \subset TP$, then σ is implemented infinitely often so no computation $\sigma \subseteq V^A_\alpha[s]$ is permanent.) Since $T^B_\alpha \subseteq T^B$, we have $T^B \not\subseteq V^A_\alpha = V^A_e$. □

Lemma 13. Each requirement S_e is satisfied.

Proof. Let α be the S_e-node on the true path. By Lemma 9 and the use of the hat-trick for U^A, we have

$$U^A \subseteq V^B_e \Leftrightarrow \lim_s m(\alpha, s) = \infty.$$

By Lemma 10, $\lim_s m(\alpha, s) < \infty$. So $U^A \not\subseteq V^B_e$. □

This completes the proof of Theorem 2. □

5. C.E. LR-degrees above low LR-degrees

In this section we show that above any low Δ^0_2 LR degree there is an incomplete c.e. LR degree. This is in contrast with the Δ^0_2 Turing degrees, in which there is a low Δ^0_2 degree which is incomparable with all intermediate c.e. degrees (the proof of this is sketched in section 5.5).

Theorem 14. Let A be a low Δ^0_2 set. There is a c.e. set B such that

$$A \leq_{LR} B <_{LR} \emptyset'.$$

Let A be a low Δ^0_2 set, given by a computable approximation $A[s]$ such that $\lim_s A(x)[s] = A(x)$ for all x. Let $\langle V_e, p_e\rangle$ be a listing of all LR-operators, and U be the second member of a universal oracle Martin-Löf test (so $\mu(U^X) \leq \frac{1}{2}$ for all $X \in 2^\omega$). We construct c.e. sets B, D and oracle Σ^0_1-classes E, H_e for all $e \in \mathbb{N}$ to satisfy the requirements:

$$\begin{aligned} N_e: &\quad H^D_e \not\subseteq V^B_e \\ R: &\quad U^A \subseteq E^B \quad\text{and}\quad \mu(E^B) < 1 \end{aligned}$$

By Theorem 1, requirement R ensures that $A \leq_{LR} B$. Since the H_e are uniformly Σ^0_1, their union $H = \cup_e H_e$ is also an oracle Σ^0_1 class. We will ensure that $\mu(H^D_e) < 2^{-e-1}$, and thus

$$\mu(H^D) \leq \sum \mu(H^D_e) \leq 2^{-e-1} < 1.$$

If N_e is satisfied for each e, then $H^D \not\subseteq V^B_e$, and $D \not\leq_{LR} B$ by Theorem 1, since $\mu(H^D) < 1$. Therefore in particular $\emptyset' \not\leq_{LR} B$.

Notice that we do not include any requirements to explicitly make $B \not\leq_{LR} A$. If desired, we could include P-requirements as in Theorem 2 to explicitly ensure $A <_{LR} B$. How to do this is discussed briefly at the end of the section. However, we may instead invoke the upward density of the c.e. LR-degrees, which follows from a result of [4], to obtain a c.e. set C with $B <_{LR} C <_{LR} \emptyset'$. Of course, if the LR-degree of A does not contain any c.e. sets then we automatically have $A <_{LR} B$ from the requirements above. (This is the interesting case since if A is $\equiv_{LR}$ to some c.e. set then we can just invoke the upward density of the c.e. LR-degrees in the first place.)

R **strategy.** Fix U as the second member of a universal oracle Martin-Löf test, so $\mu(U^A) < \frac{1}{2}$. We simply trace U^A into E^B: whenever a new computation $\rho \in U^A[s]$ appears, we put ρ into $E^B[s]$ with large use. If a B-change inadvertently removes a string ρ from $E^B[t]$ while $\rho \in U^A[t]$ is still valid, we put ρ back into E^B with the same use as previously. If an A-change invalidates the computation $\rho \in U^A[s]$, we remove ρ from E^B via a B-enumeration, if such an enumeration is not prevented by an active restraint. It is up to the N-requirements to ensure that their restraints do not prevent too many strings from being removed from E^B, so we can ensure that $\mu(E^B) < 1$.

N_e **strategy.** Recall that $\langle V_e, p_e\rangle$ is a list of all LR-operators; that is, an oracle Σ^0_1 class V_e and a dyadic rational p_e such that $\mu(V_e^X) \leq q_e$ for all $X \in 2^\omega$. We need to diagonalise against V_e, forcing measure into V_e^B without causing $\mu(H_e^D)$ to permanently increase. The basic strategy is to put a clopen set δ into H_e^D and wait until $\delta \subseteq V_e^B$. When this occurs we restrain B on the use of the computation and remove δ from H_e^D by enumerating into D. Thus $\mu(V_e^B)$ permanently increases by $\mu(\delta)$ but $\mu(H_e^D)$ does not. With a suitable choice of $\mu(\delta)$, after finitely many repetitions we will have some δ which is not covered by V_e^B, since $\mu(V_e^B)$ cannot increase above p_e. This δ will be permanently in H_e^D, and nothing else will be added to H_e^D. So $\mu(H_e^D)$ permanently increases by $\mu(\delta)$ only once.

This is complicated by the fact that B-restraints conflict with the R-strategy. Each B-restraint captures certain junk intervals in E^B, in the sense that the B-restraint will prevent the R-strategy from removing some intervals from E^B if an A-change removes them from U^A. Such strings that are in E^B but not in U^A are *junk*; we must make sure that the total junk measure captured by B-restraints is small so that $\mu(E^B) < 1$.

We can separate the N_e-requirement into finitely many subrequirements $N_{e,i}$, and assign each N-subrequirement a quota ϵ. We ask that $N_{e,i}$'s restraints contribute at most ϵ measure of junk to E^B. We will allow an N-node α to impose a restraint r at stage s only if the total junk measure that would be captured - those strings in $E^B - U^A$ with use $< r$ - is within the quota ϵ. However, the junk captured by the restraint r may later increase as the construction proceeds, as strings may be removed from U^A after the restraint is imposed. Although we can easily ensure that the restraint initially captures at most ϵ of junk, we must also ensure that the junk does not later grow too large.

This is dealt with by *measure-guessing*: we place the construction on a tree of strategies, and equip each N-node α with a backing measure-guessing node. The backing node supplies the N-node α with an approximation to $\mu(U^A)$, in the form of a rational interval $[q_\alpha, q_\alpha + \epsilon_\alpha)$. The node α works only at stages when $\mu(U^A[s]) \in [q_\alpha, q_\alpha + \epsilon_\alpha)$; that is when the 'measure guess' that $\mu(U^A) \in [q_\alpha, q_\alpha + \epsilon_\alpha)$ appears correct. By ordering the nodes with lower intervals to the left, and by applying the hat-trick to the approximation of U^A, we can ensure that the true path (the path of leftmost infinitely-often visited nodes) consists of those nodes whose measure guess is correct, ie $\mu(U^A) \in [q_\alpha, q_\alpha + \epsilon_\alpha)$ in the limit. A node may only impose a B-restraint r if the measure of junk captured by r is less than ϵ_α. If the junk captured by the B-restraint later increases by more than ϵ_α, the hat-trick applied to the approximation of U^A guarantees that the node will be reset, as in that case the approximation $\mu(U^A[s])$ drops below q_α. The result is that a node never captures more than 2ϵ much junk: ϵ much which was present when it first imposed restraint, and ϵ much which may have been added after the restraint was imposed. This technique was first used by Cholak, Greenberg and Miller [7] and later in [5] and [6].

We arrange the priority tree so that each level of the tree is occupied by a single requirement, and all the nodes of that level have the same quota. We need one additional condition to keep the junk captured by restraints under control. In a traditional tree construction, there is no bound to the number of nodes on any level of the tree that may be imposing restraint simultaneously. In our case, each such node on a particular level would potentially be contributing the same amount ϵ of junk to E^B, threatening our desire to keep $\mu(E^B) < 1$. The solution is to ensure that *at most one node on each level of the tree is imposing restraint at any time.* To satisfy this, all nodes on one level of the tree will work with the same clopen set δ, on the task of ensuring that $\delta \subseteq V_e^B$. If a node β on the same level but to the left of α has already imposed a B-restraint to preserve $\delta \subseteq V_e^B$, then α does not need to do anything since (from α's point of view) β has already satisfied $\delta \subseteq V_e^B$. If no node left of α has imposed a restraint, and α sees $\delta \subseteq V_e^B[s]$ (via a computation which does not capture too much junk), then α may impose a restraint and remove δ from H_e^D.

There is a risk here however that the junk captured by α's restraint later grows above the quota ϵ_α, and α is reset when $\mu(U^A[s])$ drops below q_α. Then we will have to start again with δ, putting δ back into H_e^D and waiting for $\delta \subseteq V_e^B[s]$ again. In fact, if α is to the right of the true path, this may happen infinitely often; in this case, $\delta \not\subseteq V_e^B$ as no computation $\delta \subseteq V_e^B[s]$ is permanent, but nor is $\delta \subseteq H_e^D$, so δ does not contribute towards requirement N_e. This problem only arises if every time a node α on level i of the tree imposes a restraint, its measure guess proves wrong and it is reset. We solve this problem by using the lowness of A. The true measure $\mu(U^A)$ is c.e. in A, and thus computable (as a real number) in A'. Since A is low, we have $\mu(U^A) \leq_T A' \leq_T \emptyset'$, and hence by the limit lemma there is a computable function $g : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ such that, for all n,

$$g(n,s) \text{ is a multiple of } 2^{-n}, \text{ for all } s; \tag{7}$$

$$\text{the limit } \hat{g}(n) = \lim_s g(n,s) \text{ exists, and} \tag{8}$$

$$\mu(U^A) \in \big[\hat{g}(n), \hat{g}(n) + 2^{-n}\big). \tag{9}$$

We will allow an $N_{e,i}$-node α to remove δ from H^D and impose B-restraint only at stages when the Π_2^0 approximation $\mu(U^A[s])$ agrees with the Δ_2^0 approximation $g(n,s)$. Since $g(n,s) = \hat{g}(n)$ for all but finitely many s, this ensures that nodes to the right of the true path take action only finitely often, and avoids the problem mentioned above. In fact, since $g(n,s)$ eventually settles, this ensures that only a single node on each level of the tree - the node with the correct measure guess - is accessible infinitely often. Thus the true path is the limit, not merely the lim inf, of the approximations to the true path TP_s.

5.1. **The priority tree and notation.** For each e, let k_e be the least number such that $2^{-k_e} < 2^{-e-2} \cdot (1 - p_e)$. Requirement N_e will use 2^{k_e} subrequirements $N_{e,i}$, $0 \leq i \leq 2^{k_e} - 1$. Fix a listing of all N subrequirements

$$N_{0,0}, \dots N_{0,2^{k_0}-1}, N_{1,0}, \dots N_{1,2^{k_1}-1}, \dots \tag{10}$$

namely, all the N_0 subrequirements in order, followed by the N_1 subrequirements, etc.

The construction takes place on a finitely branching tree, defined below, consisting of nodes labelled G or $N_{e,i}$ for some $e, i \in \mathbb{N}$ according to their length. Nodes of even length (including the root node) are labelled G; nodes of odd length $2n+1$ are labelled $N_{e,i}$ where $N_{e,i}$ is the n'th entry in the list (10). Nodes labelled $N_{e,i}$ for some e, i are referred to as N-nodes; nodes labelled $N_{e,i}$ for a fixed e are referred to as N_e-nodes. N-nodes have a single outcome 0 (a single child node $\alpha^\frown 0$ on the tree). G-nodes have four outcomes $x_0 <_L x_1 <_L x_2 <_L x_3$, corresponding to subintervals of the half-unit interval $[0, \frac{1}{2})$. Each node α is associated with an interval $[q_\alpha, q_\alpha + \epsilon_\alpha)$, where $q_\alpha, \epsilon_\alpha$ are dyadic rationals. For the root node $\emptyset$ we have $q_\emptyset = 0, \epsilon_\emptyset = \frac{1}{2}$; for other nodes the interval is defined inductively. Suppose that $[q_\alpha, q_\alpha + \epsilon_\alpha)$ is defined. If α is an N-node then it has only one child $\alpha^\frown 0$; let $q_{\alpha^\frown 0} = q_\alpha$ and $\epsilon_{\alpha^\frown 0} = \epsilon_\alpha$. If α is a G-node, then $\epsilon_{\alpha^\frown x_i} = \frac{1}{4}\epsilon_\alpha$ and $q_{\alpha^\frown x_i} = q_\alpha + i\frac{1}{4}\epsilon_\alpha$ for $0 \leq i \leq 3$; that is, we evenly subdivide $[q_\alpha, q_\alpha + \epsilon_\alpha)$ and assign the subintervals in order to $x_0 \dots x_3$. We subdivide into four to ensure that

$$\sum_{\gamma \in Z} 2\epsilon_\gamma < \epsilon_\alpha \tag{11}$$

for any set Z of nodes longer than α containing at most one node of each length. This means that, even if every subrequirement below α has a node γ imposing restraint, and capturing up to ϵ_γ of junk, the total junk is still within α's quota ϵ_α, so α will be able to act. Note that if α is a G-node then $\epsilon_\alpha = 2^{-|\alpha|-1}$ and if it is an N-node then $\epsilon_\alpha = 2^{-|\alpha|-2}$. Note also that $\mu(U^A)$ is an A-random real, and in particular is irrational, so we needn't worry about $\mu(U^A)$ lying on an endpoint of one of our rational intervals.

The ordering $<_L$ on $\{x_0, \dots x_3\}$ induces an ordering on the tree: for nodes α, β, $\alpha <_L \beta$ indicates that α is to the left of β, and $\alpha < \beta$ indicates that $\alpha <_L \beta$ or $\alpha \subset \beta$. In fact, $<_L$ agrees with the ordering of q_α's: $\alpha <_L \beta$ iff $q_\alpha < q_\beta$.

For each e, divide 2^ω evenly into 2^{k_e} many subintervals $I_{e,0}, I_{e,1} \dots I_{e,2^{k_e}-1}$. Subrequirement $N_{e,i}$ works with interval $I_{e,i}$. An $N_{e,i}$-node α pursues the following strategy. When all higher-priority N_e subrequirements are finished, α puts $I_{e,i}$ into H_e^D and waits until $I_{e,i} \subseteq V_e^B[s]$. When this occurs at a stage when the Π_2^0 approximation $\mu(U^A[s])$ agrees with the Δ_2^0 approximation g, it restrains B on the use of this computation to preserve $I_{e,i} \subseteq V_e^B$ (if the restraint does not capture too much measure in E^B) and removes $I_{e,i}$ from H_e^D by enumerating into D. Henceforth, as long as α is not injured due to the measure $\mu(U^A[s])$ decreasing, we will have $I_{e,i} \subseteq V_e^B$ and have succeeded in increasing $\mu(V_e^B)$ by 2^{-k_e}.

Each N-node α has a parameter r_α which is the restraint that α wishes to impose on B. Let $R_\alpha[s] = \max_{\beta<\alpha} r_\beta[s]$ be the total restraint imposed by N-nodes of higher priority than α.

As in Theorem 2, we assume that when a string is put into any H_e^D with some use u, it remains there until the number u is explicitly enumerated into D. This assumption does not apply to E^B however. Each time we put a string into $H_e^D[s]$ it will be while carrying out the instructions for some N-node. If an interval I is in $H_e^D[s]$, then we say that I *is in* $H_e^D[s]$ *on account of* α if I was most recently put into H_e^D while carrying out the instructions for α.

We again use the hat-trick for the enumeration of U^A. Let $a_0 = 0$, and for $s > 0$ let a_s be the least number such that $A(a_s)[s] \neq A(a_s)[s-1]$, or $a_s = s$ if such number does not exist. Let

$$\widehat{U^A}[s] = U^{A\restriction a_s}[s] = \{\sigma : \sigma \text{ is in } U^A[s] \text{ with use} \leq a_s\}.$$

Henceforth we omit the hat and write $U^A[s]$ to mean $\widehat{U^A}[s]$. In this case, the hat-trick ensures that there are infinitely many true stages, at which $U^A[s] \subseteq U^A$ and $\mu(U^A[s]) \leq \mu(U^A)$.

In the construction, we will explicitly define the approximation to the true path TP_s. When we take action for a node on TP_s, we will stop defining TP_s, so TP_s will not always have length s. This means that we cannot rely on a node α being reset due to $\mathrm{TP}_s <_L \alpha$ when its measure guess becomes wrong. Hence at each stage we must explicitly reset all nodes whose measure guess has become wrong. For convenience, we do this resetting only at even stages of the construction, and perform the other tasks of the construction only at odd stages. We can assume that we are given approximations of A, U, V_i etc that change only on even stages. That is, $A[2s] = A[2s+1]$ for all s, and similarly for U, V_i (as sets of axioms).

To *reset* an $N_{e,i}$-node α at stage $s+1$ means to set $r_\alpha[s+1] = 0$, and if $I_{e,i}$ is in $H_e^D[s]$ on account of α then remove $I_{e,i}$ from $H_e^D[s+1]$ by enumerating its use into $D[s+1]$.

5.2. **The construction.** Initially $B[0] = D[0] = H[0] = \emptyset$ and $r_\alpha[0] = 0$ for all N-nodes α. At stage $s+1$ we are given $A[s], B[s]$ etc and any changes we make are in order to define $B[s+1]$ etc.

At stage $s+1$ where $s+1$ is even, reset any N-nodes α such that $\mu(U^A[s]) < q_\alpha$ and $r_\alpha[s] \neq 0$.

At stage $s+1$ where $s+1$ is odd, perform steps 1 and 2 in order.

Step 1. We define the approximation to the true path TP_{s+1} and take action for some node on TP_{s+1}. Suppose inductively that $\mathrm{TP}_{s+1} \restriction n$ is defined, for $n \geq 0$. If $n = s+1$ then stop defining TP_{s+1} and go to step 2. Otherwise let $\alpha = \mathrm{TP}_{s+1} \restriction n$ and go to the appropriate case below.

• α **is a G-node**. Inductively, $\mu(U^A[s]) \in [q_\alpha, q_\alpha + \epsilon_\alpha)$. Let i be such that $\mu(U^A[s]) \in [q_{\alpha^\frown x_i}, q_{\alpha^\frown x_i} + \epsilon_{\alpha^\frown x_i})$. If $g(n+3, s) = q_{\alpha^\frown x_i}$ then let $\mathrm{TP}_{s+1} \restriction n+1 = \alpha^\frown x_i$ and continue defining TP_{s+1}. If not then stop defining TP_{s+1} and go to step 2.

• α **is an $N_{e,i}$-node for some** $e, i \in \mathbb{N}$. Say that α is *active* at stage $s+1$ if $\forall j < i$ there is an $N_{e,j}$-node $\beta < \alpha$ with $r_\beta[s] \neq 0$. Go to the least case below which holds.

(I) *Higher priority subrequirements have finished and α is ready to start.* α is active; there is no $N_{e,i}$-node $\alpha' \leq_L \alpha$ with $r_{\alpha'}[s] \neq 0$; and $I_{e,i} \notin H^D_e[s]$. Then put $I_{e,i}$ into $H^D_e[s+1]$ with large use. Stop defining TP_{s+1} and go to step 2.

(II) *α's interval $I_{e,i}$ has appeared in $V^B_e[s]$ with a believable computation and we are ready to restrain B.* α is active; there is no $N_{e,i}$-node $\alpha' \leq_L \alpha$ with $r_{\alpha'}[s] \neq 0$; $I_{e,i} \in H^D_e[s]$ and $I_{e,i} \subseteq V^B_e[s]$ with use u such that

$$\mu(E^{B \restriction u}[s] - E^{B \restriction R_\alpha}[s] - U^A[s]) < \epsilon_\alpha. \tag{12}$$

Then set $r_\alpha[s+1] = u$. Remove $I_{e,i}$ from $H^D_e[s+1]$ by enumerating the use into $D[s+1]$. Reset all N-nodes of lower priority than α, stop defining TP_{s+1} and go to step 2.

(III) Otherwise, set $\mathrm{TP}_{s+1} \restriction n+1 = \alpha^\frown 0$, the unique child of α, and continue defining TP_{s+1}.

Step 2. Let $R = \max_\alpha r_\alpha[s+1]$ be the total restraint imposed by all nodes after step 1. Enumerate $R+1$ into $B[s+1]$ to remove some junk intervals from $E^B[s] - U^A[s]$.

End of construction.

5.3. **Verification.** The first lemma verifies the consistency between restraints and intervals in H^D.

Lemma 15. Let α be an $N_{e,i}$-node and s a stage. If $r_\alpha[s] \neq 0$ then $I_{e,i} \notin H^D_e[s]$.

Proof. Suppose for contradiction that $r_\alpha[s] \neq 0$ and $I_{e,i} \in H^D_e[s]$ on account of some node β. Let s_1 be the greatest stage $\leq s$ at which r_α was set non-zero and s_2 be the greatest stage $\leq s$ at which $I_{e,i}$ was put into $H^D_e[s_2]$ by β. Suppose first that $s_1 < s_2$. We cannot have $\beta \geq \alpha$ as (I) cannot hold for β while $r_\alpha[s_2] \neq 0$; nor can we have $\beta < \alpha$ as then α would be reset and r_α set to 0 at s_2 which contradicts $r_\alpha[s] \neq 0$. So $s_1 < s_2$ is impossible. Suppose then that $s_1 > s_2$. If $\alpha < \beta$ then β would be reset at s_1 and $I_{e,i}$ removed from $H^D_e[s_1]$, contradicting $I_{e,i} \in H^D_e[s]$ on account of β. If $\alpha \geq \beta$ then $I_{e,i}$ would be removed from $H^D_e[s_1]$ by (II), again contradicting $I_{e,i} \in H^D_e[s]$. Finally, $s_1 = s_2$ is impossible as at most one of actions (I), (II) is taken at any stage. □

We next verify that $H^D_e[s]$ contains at most one of the intervals $I_{e,i}$ at any time, and thus $\mu(H^D) < 1$.

Lemma 16. For all e and at any stage s, either $H^D_e[s] = \emptyset$ or $H^D_e[s] = I_{e,i}$ for some i.

Proof. Suppose on the contrary, that $H^D_e[s]$ contains both $I_{e,i}$ and $I_{e,j}$ for some $i \neq j$. Suppose that $I_{e,i}$, $I_{e,j}$ were added to $H^D_e[s]$ at stage s_0, s_1 by N_e-node α, β, respectively. Note that at most one interval is added to H^D_e at each stage (since the stage is ended if (I) holds); thus we may assume that $s_0 < s_1$. We consider the possibilities for the position of α relative to β.

If $\beta <_L \alpha$, then $\mu(U^A[s_1]) < q_\alpha$, and α would have been reset at the (even) stage $s_1 - 1$. Thus $I_{e,i}$ could not be in $H^D_e[s_1]$.

If $\beta \geq \alpha$ and $|\beta| > |\alpha|$, then β must satisfy (I) at s_1; in particular, there must be an $N_{e,i}$-node $\alpha' \leq \beta$ with $r_{\alpha'}[s_1] \neq 0$. By Lemma 15, $I_{e,i} \notin H^D_e[s_1]$, contradicting $I_{e,i} \in H^D_e[s]$.

Finally, $|\beta| < |\alpha|$ and $\beta \not<_L \alpha$. In this case, at s_0 there is some $N_{e,j}$-node $\beta' \leq \alpha$ with $r_{\beta'}[s_0] \neq 0$. Since β satisfies (I) at $s_1 > s_0$, the node β' must have been reset at some s', $s_0 < s' < s_1$. But then α would be reset at s' also, and $I_{e,i}$ would have been removed from $H^D_e[s'+1]$. □

Since $H^D = \cup_e H_e^D$, it follows that $\mu(H^D) \leq \sum_e \mu(H_e^D) \leq 2^{-k_e} < 1$.

Next we verify that each restraint does indeed capture no more than its quota of junk. For an N-node α and $s > 0$, let

$$J_\alpha[s] = E^{B\restriction r_\alpha}[s] - E^{B\restriction R_\alpha}[s] - U^A[s-1]$$

be the junk intervals restrained by α at the end of stage s.

Lemma 17. For any N-node α and for any even s,

$$\mu(J_\alpha[s]) \leq 2\epsilon_\alpha.$$

Proof. Suppose that $r_\alpha[s] \neq 0$, and let $t+1$ be the greatest stage $\leq s$ when r_α was set nonzero. Write $r = r_\alpha[s]$. If new strings are added to E^B after t then they are added with fresh use, and if $B[s] \restriction r \neq B[t] \restriction r$ then α would be reset between $t+1$ and s. Thus $E^{B\restriction r}[s] = E^{B\restriction r}[t]$. Also, $R_\alpha[t] = R_\alpha[s]$ as otherwise α would have been reset. So,

$$\begin{aligned}\mu(J_\alpha[s]) &= \mu(E^{B\restriction r}[s] - E^{B\restriction R_\alpha}[s] - U^A[s-1]) \\ &\leq \mu(E^{B\restriction r}[t] - E^{B\restriction R_\alpha}[t] - U^A[t]) + \mu(U^A[t] - U^A[s-1]).\end{aligned} \tag{13}$$

The first term of (13) is the junk that was captured by α when it imposed its restraint; the second is that which becomes junk after the restraint was imposed. By (12), the first term is less than ϵ_α. Suppose that $\mu(U^A[t] - U^A[s-1]) \geq \epsilon_\alpha$. But then by the hat-trick there would be an even stage t' with $t < t' \leq s$ such that $\mu(U^A[t']) \leq \mu(U^A[t]) - \epsilon_\alpha \leq q_\alpha$. At t', α would be reset, contradicting the definition of t. □

Since we sometimes end the definition of TP_s before it reaches length s, we must verify that the true path of infinitely often visited nodes exists. The following lemma verifies that the true path exists, is infinite, and that the N-nodes on the true path reach a limit state.

Lemma 18. For every n there is a unique node α of length n which is accessible infinitely often. Moreover, α is accessible at all but finitely many true stages, $\mu(U^A) \in [q_\alpha, q_\alpha + \epsilon_\alpha)$, and if α is an $N_{e,i}$-node then α is reset only finitely often, the definition of TP_s is ended at α only finitely often, and exactly one of the following hold:

(i) there is a $j \leq i$ and an $N_{e,j}$-node $\beta \leq \alpha$ and a stage t such that $I_{e,j} \in H_e^D$ permanently on account of β after t and $I_{e,j} \not\subseteq V_e^B$;

(ii) there is an $N_{e,i}$-node $\beta \leq_L \alpha$ and a stage t such that $r_\beta[t] \neq 0$, β is not reset after t, $I_{e,i} \subseteq V_e^B$ and $I_{e,i} \notin H_e^D[s]$ at any $s \geq t$.

Proof. We use induction on n. Certainly the claim is true for the root node (which is a G-node). Assume that the claim is true for n, and let β be the unique node of length n which is accessible at infinitely many stages including all but finitely many true stages. Consider first the case that β is an N-node. In this case β has only a single child $\alpha = \beta^\frown 0$, with $q_\alpha = q_\beta$, $\epsilon_\alpha = \epsilon_\beta$, and since the definition of TP_s is ended at β only finitely often, α is accessible at all but finitely many stages when β is accessible.

For the remainder of the proof we consider the case that β is a G-node. Inductively we have that β is accessible at every true stage after some s_0 and that $\mu(U^A) \in [q_\beta, q_\beta + \epsilon_\beta)$. Hence there is an i such that $\mu(U^A) \in [q_{\beta^\frown x_i}, q_{\beta^\frown x_i} + \epsilon_{\beta^\frown x_i})$. Let $\alpha = \beta^\frown x_i$. By (7)-(9), there is a stage $s_1 \geq s_0$ such that $g(n+3, s) = q_\alpha$ for all $s > s_1$. Since $\mu(U^A) > q_\beta$, there is a u and $s_2 \geq s_1$ such that $\mu(U^{A\restriction u}) > q_\beta$ and $A[s] \restriction u = A \restriction u$ for all $s > s_2$. At every true stage $s > s_2$ we have $\mu(U^A[s]) \in [q_\alpha, q_\alpha + \epsilon_\alpha)$ and $g(n+3, s) = q_\alpha$, so α will be accessible. No other node α of length $n+1$ will satisfy $g(n+3, s) = q_{\alpha'}$ after s_1, so α is the only node of length $n+1$ which is accessible infinitely often.

Next we verify that α is reset only finitely often. As argued above, $\mu(U^A[s]) < q_\alpha$ for only finitely many s, so α is reset at even stages only finitely often. The other way that α can be reset is if (II) holds for some N-node $\alpha' < \alpha$. This can happen only finitely often for nodes $\alpha' <_L \alpha$ as such nodes are accessible only finitely often; by induction (II) can hold only finitely often for nodes $\alpha' \subset \alpha$ as the definition of TP_s is ended when this occurs. So α is reset only finitely often.

Next we show that α satisfies a limit state, satisfying (i) or (ii). If some N_e-node $\gamma \subset \alpha$ satisfies (i) then α also satisfies (i). Otherwise, suppose that every N_e-node $\gamma \subset \alpha$ satisfies (ii), and let t_0 be an α-stage such that α is not reset after t_0, no node $\delta >_L \alpha$ is accessible after t_0, and all N_e-nodes $\gamma \subset \alpha$ satisfy (ii) by t_0. If some $N_{e,i}$-node $\alpha' <_L \alpha$ has $r_{\alpha'}[t_0] \neq 0$ then that restraint is permanent and α satisfies (ii) due to α'. Suppose then that $r_{\alpha'}[t_0] = 0$ for all $N_{e,i}$-nodes $\alpha' <_L \alpha$. If $I_{e,i} \notin H_e^D[t_0]$ then it will be added to H_e^D at t_0; if $I_{e,i} \in H_e^D[t_0]$ then it must be due to some node $\alpha' <_L \alpha$ as all nodes $>_L \alpha$ were reset at the even stage $t_0 - 1$. Either way, $I_{e,i} \in H_e^D[t_0+1]$ due to a node $\alpha' \leq_L \alpha$. Since no such node gets reset after t_0, and no node $\delta >_L \alpha$ is accessible after t_0, $I_{e,i}$ can only be removed from H_e^D after t_0 if (II) holds for α. If this happens then α will impose a permanent restraint and will satisfy (ii). It suffices now to show that if $I_{e,i} \subseteq V_e^B$ then eventually the measure condition (12) is satisfied and (II) will hold.

Suppose that $I_{e,i} \subseteq V_e^B$ with use v, and let t_1 be the second α-stage after t_0 such that

$$B[t_1] \upharpoonright v = B \upharpoonright v \ \text{ and } \ E^{B\upharpoonright v}[t_1] \cap U^A[t_1] = E^{B\upharpoonright v} \cap U^A \text{ (as sets of strings)}$$

(such t_1 exists because of the hat-trick). Every string in $E^{B\upharpoonright v}[t_1] - E^{B\upharpoonright R_\alpha}[t_1] - U^A[t_1]$ is in $J_\gamma[t_1]$ for some $\gamma > \alpha$; as otherwise it would be removed in step 2 of the construction contradicting the choice of t_1. Let

$$Z = \{\gamma : \gamma > \alpha \text{ and } r_\gamma[t_1] \neq 0\}$$

be the lower-priority nodes with nonzero restraint at t_1. Then

$$\begin{aligned} \mu\big(E^{B\upharpoonright v}[t_1] - E^{B\upharpoonright R_\alpha}[t_1] - U^A[t_1]\big) &\leq \sum_{\gamma \in Z} \mu(J_\gamma[t_1]) \\ &\leq \sum_{\gamma\in Z} 2\epsilon_\gamma \\ &\leq \epsilon_\alpha \end{aligned}$$

by Lemma 17 and (11). Thus eventually (12) is satisfied, so if $I_{e,i} \subseteq V_e^B$ then α satisfies (i).

Finally we verify that the definition of TP_s is ended at α only finitely often. This happens only when (I) or (II) hold for α. Suppose that (i) holds for α. Then (II) cannot hold after t, as if it did, the restraint r_α would be respected and $I_{e,i} \subseteq V_e^B$ in contradiction to (i). If $j = i$ then (I) cannot hold after t as $I_{e,i} \in H_e^D$ already. If $j < i$ then by Lemma 15 α is not active after t so (I) cannot hold. Suppose that (ii) holds for α; then it is immediate from the construction that neither (I) nor (II) can hold after t. Hence the definition of TP_s is ended at α only finitely often. □

Having established that each N-node reaches a limit state, we can easily verify that each requirement is satisfied.

Lemma 19. For all e there is an $I_{e,i}$ such that $I_{e,i} \in H_e^D$ and $I_{e,i} \not\subseteq V_e^B$. Thus each requirement N_e is satisfied.

Proof. If some $N_{e,i}$-node on the true path satisfies (i) of Lemma 18, then N_e is satisfied by $I_{e,i}$ for the least such i. To see that there is always such an i for each e, note that if not, every N_e-node on TP must satisfy (ii), and thus $I_{e,i} \subseteq V_e^B$ for each i. But there are 2^{k_e} many such $I_{e,i}$, each has measure 2^{-k_e}, and they are pairwise disjoint. Then $\mu(V_e^B) \geq 2^{k_e} \cdot 2^{-k_e} = 1$, which contradicts $\mu(V_e^B) \leq q_e < 1$. □

Finally we verify that requirement R is satisfied.

Lemma 20. $A \leq_{LR} B$.

Proof. By the definition of E, once an interval appears in U^A via a permanent computation it will henceforth always be in E^B with the same use. Thus $U^A \subseteq E^B$. We must verify that $\mu(E^B) < 1$. Since $\mu(U^A) \leq \frac{1}{4}$, it suffices to show that

$$\mu(E^{B\restriction n}[s] - U^A[s]) \leq \frac{1}{2}$$

for all $n \in \mathbb{N}$ and sufficiently large s. Fix n and let s_0 be a stage such that

$$B \restriction n[s_0] = B \restriction n \text{ and } E^{B\restriction n}[s_0] - U^A[s_0] = E^{B\restriction n} - U^A$$

(as sets of strings). Then for all $s \geq s_0$ we have

$$E^{B\restriction n}[s] - U^A[s] \subseteq \bigcup_{\alpha} J_\alpha[s],$$

and by Lemma 17 and the fact that at any time there is at most one node of each length with nonzero restraint, $\mu(E^{B\restriction n}[s] - U^A[s]) \leq \sum_e 2^{-e-2} = \frac{1}{2}$. □

This completes the proof of Theorem 14. □

The key aspect of this construction is the fact that *leftward movement of TP_s depends only on the approximation of A*. That is, if α is accessible at t, and later $\mathrm{TP}_s <_L \alpha$ for $s > t$, then there must have been an A-change between s and t which removed some strings from U^A and made the approximation to $\mu(U^A)$ decrease. This is the fact that allows us to use g to limit the movement of TP_s in order to prevent interference with α's strategy. This seems to be a significant limitation on the technique; it is unclear how, if at all, this construction could be combined with other requirements which involve branching on the tree of strategies that does not depend solely on A, for instance minimal pair or non-cupping-style requirements in which the outcomes depend on convergence of computations rather than on the approximation to A. However, it can be combined with the P-strategy of Theorem 2. Although in the general case the P-strategy causes the construction to move to the left independently of A-changes, for instance when an attack is implemented for the first time, in the case when A is low we can use the Robinson guessing technique as described at the end of section 4.4 to reduce the P-strategy to a finite-injury procedure. This procedure is entirely compatible with the construction above.

5.4. **The N-strategy with A LR-incomplete.** It was originally hoped that the construction of Theorem 14 could be combined with the LR-incompleteness strategy of section 3 to show that every incomplete Δ^0_2 LR-degree is bounded by an incomplete c.e. LR-degree; that is, that above any LR-incomplete Δ^0_2 set there is an LR-incomplete c.e. set. Unfortunately there are obstacles to performing the construction in this most general case. Indeed, it is not known if every incomplete Δ^0_2 LR-degree is bounded by an incomplete c.e. LR-degree. We note though that the construction of Theorem 14 can be combined with the strategy of section 3 in the case when the set A is c.e. (rather than Δ^0_2) and LR-incomplete (rather than low). This can give an alternate proof of the upwards density of the c.e. LR-degrees (which was previously proved in [4]). Though such a proof may be of little interest itself, the technique might be useful for other constructions in the LR-degrees. We refer the reader to [14] for a discussion of the construction in the case when A is c.e. and LR-incomplete, and of the obstacles to the more general case when A is Δ^0_2 and LR-incomplete.

5.5. **Differences with the Turing degrees.** Theorem 14 shows a contrast between the c.e. and Δ^0_2 Turing degrees and the c.e. and Δ^0_2 LR-degrees. Yates [20] constructed a Δ^0_2 Turing degree which is incomparable with all c.e. Turing degrees except $\mathbf{0}$ and $\mathbf{0}'$. Yates used an oracle construction with a $\emptyset'$ oracle to construct the required Δ^0_2 set A. It is possible to adapt his construction to also ensure that A is low (the proof is sketched below). Hence there is a low Δ^0_2 Turing degree that is incomparable with all intermediate c.e. Turing degrees. In particular, it has no incomplete c.e. Turing degree above it. This is in contrast to Theorem 14 which shows that every low Δ^0_2 LR-degree is bounded by an LR-incomplete c.e. LR-degree.

It is possible to further adapt Yates's construction to make the set A be low and non-low-for-random in addition to being incomparable with all intermediate c.e. Turing degrees. We note that if we apply Theorem 14 to such an A, we obtain a c.e. set B such that $A <_{LR} B$ but $A \not\leq_T B$. Hence the construction of Theorem 14 does not automatically produce $B \geq_T A$.

We sketch the construction of a Δ^0_2 set which is low, non-low-for-random and Turing incomparable with all c.e. sets of intermediate Turing degree. We build A via finite extensions $\alpha_0 \subseteq \alpha_1 \ldots$, so $A = \cup_s \alpha_s$. We have a listing W_e of all c.e. sets, and the W_e are uniformly computable from $\emptyset'$. Φ_e is a standard listing of Turing functionals. To make A low we use the usual technique of 'forcing the jump': at stage s if there is a string $\tau \supset \alpha_s$ such that $\Phi^\tau_e(e)\downarrow$, then make $A \supset \tau$ by suitable choice of α_{s+1}. The strategy for making $W_e \not\leq_T A$ if W_e is noncomputable is the usual strategy of looking for splittings of Φ_i for each i. To make $A \not\leq_T W_e$ if $\emptyset' \not\leq_T W_e$, we utilise a function $f \leq_T \emptyset'$ such that f is not dominated by any function of degree $< \mathbf{0}'$. We use a bounded search, bounded at stage s by $f(s)$, to search for an x such that $A(x)$ is not yet defined and $\Phi^{W_e}_i(x)\downarrow$. We can then define $A(x) \neq \Phi^{W_e}_i(x)$. Because we use only a bounded search, we must place the requirements in a finite injury setting, to allow a higher priority requirement to take action as soon as its bounded search finds a suitable candidate. This is straightforward and may be found in [17] XI.3.6.

The only remaining requirement is to make A non-low-for-random. Fix a member U of a universal oracle Martin-Löf test, and a listing $\langle V_i, p_i\rangle$ of Σ^0_1 classes along with a rational bound $p_i < 1$ such that $\mu(V_i) \leq p_i$. We can assume that the set U^τ is finite and uniformly computable from $\tau \in 2^{<\omega}$. Given any $\alpha \in 2^{<\omega}$, we want to find an extension $\alpha' \supset \alpha$ such that $U^{\alpha'} \nsubseteq V_i$. Using the $\emptyset'$ oracle we can search for an α' and a string σ such that

$$\sigma \in U^{\alpha'} \text{ and } \sigma \nsubseteq V_i$$

and make $A \supset \alpha'$. Such an α' and σ are guaranteed to exist: let $X \in 2^\omega$ be such that $X \notin V_i$; such X exists since $\mu(V_i) < 1$. There is a $Z \supset \alpha$ such that X is not random relative to Z (for instance, a Z such that $X \leq_T Z$). Since U is a universal test, there is a σ and an n such that $\sigma \subset X$ and $\sigma \in U^{Z\restriction n}$. The string $\alpha' = Z \restriction n$ and σ are as required. We can combine this with the previous strategies in a finite injury setting using a $\emptyset'$ oracle to construct the set A.

References

[1] G. Barmpalias. Elementary differences between the degrees of unsolvability and the degrees of compressibility. *Annals of Pure and Applied Logic* **161** (2010), 923–934.

[2] G. Barmpalias and A. E. M. Lewis. Chaitin's halting probability and the compression of strings using oracles. *Proceedings of the Royal Society A* **467** (2011), 2912–2926.

[3] G. Barmpalias, A. E. M. Lewis and M. Soskova. Randomness, lowness and degrees. *Journal of Symbolic Logic* **73** (2008), 559–577.

[4] G. Barmpalias, A. E. M. Lewis and F. Stephan. Π^0_1 classes, LR degrees and Turing degrees. *Annals of Pure and Applied Logic* **156** (2008), 21–38.

[5] G. Barmpalias and A. Montalbán. A cappable almost everywhere dominating computably enumerable degree. *Electronic Notes in Theoretical Computer Science* **167** (2007).

[6] G. Barmpalias and A. Morphett. Non-cupping, measure and computably enumerable splittings. *Mathematical Structures in Computer Science* **19** (2009), 25–43.

[7] P. Cholak, N. Greenberg and J. S. Miller. Uniform almost everywhere domination. *Journal of Symbolic Logic* **71** (2006), 1057–1072.

[8] D. Diamondstone. Low upper bounds in the LR degrees. *Annals of Pure and Applied Logic*, to appear.

[9] B. Kjos-Hanssen. Low for random reals and positive-measure domination. *Proceedings of the American Mathematical Society* **135** (2007), 3703–3709.

[10] B. Kjos-Hanssen, J. S. Miller and D. R. Solomon. Lowness notions, measure and domination. *Journal of the London Mathematical Society* **85** (2012), to appear.

[11] A. Kučera and S. A. Terwijn. Lowness for the class of random sets. *Journal of Symbolic Logic* **64** (1999), 1396–1402.
[12] J. S. Miller. The K-degrees, low for K degrees, and weakly low for K sets. *Notre Dame Journal of Formal Logic* **50** (2010), 381–391.
[13] J. S. Miller and L. Yu. Unpublished notes (2006).
[14] A. Morphett. *Degrees of Computability and Randomness*. PhD. thesis, University of Leeds, 2009.
[15] A. Nies. *Computability and Randomness*. Oxford University Press, Oxford, 2009.
[16] A. Nies. Lowness properties and randomness. *Advances in Mathematics* **197** (2005), 274–305.
[17] P. G. Odifreddi. *Classical Recursion Theory Volume II*. North-Holland, Amsterdam, Oxford, 1999.
[18] G. E. Sacks. On the degrees less than $\mathbf{0}'$. *Annals of Mathematics* **77** (1963), 221–231.
[19] R. I. Soare. *Recursively Enumerable Sets and Degrees*. Springer-Verlag, Berlin, London, 1987.
[20] C. E. M. Yates. Recursively enumerable degrees and the degrees less than $0'$. In *Sets, Models and Recursion Theory (Proc. Summer School Math. Logic and Tenth Logic Colloq., Leicester, 1965)*, North-Holland, Amsterdam, 1967, 264–271.

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