NON-CUPPING, MEASURE AND COMPUTABLY ENUMERABLE SPLITTINGS

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Abstract. We show that there is a computably enumerable function f (i.e. computably approximable from below) which dominates almost all functions and $f \oplus W$ is incomplete, for all incomplete computably enumerable sets W. Our main methodology is the LR equivalence relation on reals: $A \equiv_{LR} B$ iff the notions of A-randomness and B-randomness coincide. We also show that there are c.e. sets which cannot be split into two c.e. sets of the same LR degree. Moreover a c.e. set is low for random iff it computes no c.e. set with this property.

1. Introduction

Computability theory studies the real line from the point of view of relative computation. Interactions with measure theory were explored from fairly early on, see for example [7, 18, 22]. The study of algorithmic randomness has produced a large body of work on measure and computability; good references for this are Downey and Hirshfeldt [9] and Nies [21]. More recently, Dobrinen and Simpson [8] introduced the notion of almost everywhere domination which was investigated more deeply in several follow-up papers [3, 13, 5, 12]. A function f is *almost-everywhere dominating* if $\mu\{X \in 2^\omega : f \text{ dominates all total } g \leq_T X\} = 1$ where μ is the Lebesgue measure, and a set is almost-everywhere dominating if it computes such a function f. This notion is degree-theoretic and we can also talk about almost everywhere dominating degrees. In this paper we are interested in the computably enumerable almost everywhere dominating degrees. Nies [20] noticed that these degrees $\mathbf{a}$ are high, i.e. $\mathbf{a}' \geq \mathbf{0}''$, and Binns, Kjos-Hanssen, Lerman and Solomon showed that there are high c.e. degrees which are not almost everywhere dominating. Cholak, Greenberg and Miller [5] established the existence of incomplete c.e. almost everywhere dominating degrees, and Barmpalias and Montalban [3] showed that some of them are

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halves of minimal pairs. In section 4 we show that some of these c.e. degrees are non-cuppable, i.e. their join with any incomplete c.e. degree is incomplete.

Theorem 1.1. There is a c.e. almost everywhere dominating set A such that $A \oplus W \not\equiv_T \emptyset'$ for all c.e. $W <_T \emptyset'$.

Theorem 1.1 has the very interesting corollary that if a set is computed by all almost everywhere dominating c.e. degrees, then it must be non-cuppable (the existence of noncomputable such sets is still open). Also, it can be viewed as a generalization of a theorem of Harrington (see [19]) which asserts that there is a function of c.e. degree which dominates all computable functions and has incomplete join with all incomplete c.e. sets. A fundamental question, which also served as motivation for Theorem 1.1 is whether almost everywhere dominating sets have degree-theoretic properties which are not shared by all the high degrees. More precisely, is there a formula ϕ in the language of $(\mathcal{R}, <)$ (where $\mathcal{R}$ denotes the c.e. Turing degrees) such that for all $\mathbf{a}$ c.e. almost everywhere dominating $\phi(\mathbf{a})$ holds but there is a high c.e. degree $\mathbf{b}$ such that $\phi(\mathbf{b})$ fails?

In section 3 we consider splittings of c.e. sets in relation with relative randomness. It is a natural question whether a c.e. set is the disjoint union of two c.e. sets B, C which induce the same notion of randomness (i.e. the class of random numbers relative to A is the same as the class of random numbers relative to B). We show that this is not always the case and that a set is low for random iff it can compute such a counterexample.

2. Preliminaries

In the following, we use c.e. sets of strings to generate subclasses of the Cantor space. In particular, we never use the relations $\subset$, $\subseteq$, $\supset$ and $\supseteq$, the measure μ and the operations $\cap$ and $\cup$ for sets U of strings; these relations and operations always refer to the class $S(U) = \{A \in 2^\omega \mid \exists n(A \upharpoonright n \in U)\}$. In other words, $\mu(U)$ is $\mu(S(U))$, $U \subseteq V$ iff $S(U) \subseteq S(V)$ and $U \cap V$ denotes actually $S(U) \cap S(V)$, not $S(U \cap V)$. For union, $S(U \cup V)$ and $S(U) \cup S(V)$ would, for both interpretations of $\cup$, anyway be the same. We recall some basic notions of relative randomness. An oracle Martin-Löf test (U_e) is a uniform sequence of oracle machines which output finite binary strings such that if U_e^B is the range of the e-th machine with oracle $B \in 2^\omega$ then for all $B \in 2^\omega$, $e \in \mathbb{N}$, $\mu(U_e^B) < 2^{-(e+1)}$ and $U_e^B \supseteq U_{e+1}^B$. A real A is called B-random if for every oracle Martin-Löf test (U_e) we have $A \notin \cap_e U_e^B$. A universal oracle Martin-Löf test is an oracle Martin-Löf test (U_e) such that for every $A, B \in 2^\omega$, A is B-random iff $A \notin \cap_e U_e^B$. Given any oracle Martin-Löf test (U_e), each U_e can be thought of as a c.e. set of axioms $\langle \tau, \sigma \rangle$. If $B \in 2^\omega$ then $U_e^B = \{\sigma \mid \exists \tau(\tau \subset B \ \wedge \ \langle \tau, \sigma \rangle \in U_e)\}$. The suffix $[s]$ indicates the value of a parameter at the beginning of stage s. The notion of almost everywhere domination turned out to be very related with the so-called LR reduciblity, defined by Nies [20]. We say that a set A is LR reducible to set B (and write $A \leq_{LR} B$) if all B-random reals are also A-random. Kjos-Hanssen, Miller and Solomon [13] (also see [23]) showed that A is almost everywhere dominating iff $\emptyset' \leq_{LR} A$. Kjos-Hanssen [12] showed that $A \leq_{LR} B$ iff for some member U of a universal oracle Martin-Löf test, there is a $\Sigma^0_1(A)$ class V^A with $U^B \subseteq V^A$ and $\mu\big(V^A\big) < 1$.

3. Splittings of computably enumerable sets inside their LR-degree

Given a c.e. set it is natural to ask if it can be expressed as the disjoint union of two c.e. sets of the same degree as itself. In the context of Turing degrees this notion has been widely studied. Lachlan [14] showed that not every c.e. set has this property. The c.e. sets which can be split into two (disjoint) c.e. sets of the same degree are known as *mitotic*. Ladner [15, 16] studied further this notion, showing that every noncomputable c.e. set computes a non-mitotic set and that there is a non-zero Turing degree whose c.e. sets are all mitotic. More results about this notion were shown in [10], and the reader can find a comprehensive survey on the general theme of splittings of c.e. sets in [11].

It is interesting to carry such notions in the context of the LR reducibility. If a c.e. set is low for random, then obviously it can be split into two c.e. sets of the same LR degree. However we show that there is a c.e. set (even a complete one) which does not have this property. That is, there is a c.e. set which cannot be expressed as a disjoint union of two c.e. sets B, C such that the class of B-random numbers is the same as the class of C-random numbers. Moreover, we show that every c.e. set which is not low for random computes a c.e. set which cannot be split into two c.e. sets of the same LR degree. The latter construction is interesting as it demonstrates a notion of "non-low for random permitting": c.e. sets which are not low for random permit certain properties to occur in the Turing degrees below them, as this happens with noncomputable, array noncomputable, non-low$_2$ sets etc.

Theorem 3.1. There is a c.e. set A that cannot be split into two c.e. sets X, Y such that $A \equiv_{LR} X \equiv_{LR} Y$. Moreover A can be such that $A \equiv_T \emptyset'$.

Proof. Let (X_i, Y_i, V_i, q_i) be an effective list of all quadruples (X, Y, V, q) of c.e. sets X, Y with $X \cap Y = \emptyset$ and pairs V, q where V is a c.e. operator such that $\mu(V^\beta) < q$ for all $\beta \in 2^\omega$, and $q < 1$. It suffices to construct a c.e. set A and a uniform sequence (T^A_e) of $\Sigma^0_1(A)$ classes such that $\mu(T^A_e) < 2^{-e-1}$ and the following requirements are satisfied:

$$R_i : X_i \cup Y_i = A \Rightarrow T^A_i \not\subseteq V^{X_i}_i \text{ or } T^A_i \not\subseteq V^{Y_i}_i$$

(the construction will automatically satisfy $A \equiv_T \emptyset'$). Then if $T^A = \cup_i T^A_i$ we have $\mu(T^A) < 1$ and for every $i \in \mathbb{N}$, if $X_i \cup Y_i = A$ then either $T^A \not\subseteq V^{X_i}_i$ or $T^A \not\subseteq V^{Y_i}_i$, which is what we wanted. For each i we define the quota $p_i := (1-q_i)\cdot 2^{-i-2}$ for R_i. The idea for the satisfaction of R_i is to put a clopen set

$B_i \subseteq 2^\omega - V_i^{X_i}$ of measure $p_i/2$ and a clopen set $D_i \subseteq 2^\omega - V_i^{Y_i}$ of measure $p_i/2$ into T_i^A (with use u_i), and wait until $C_i \subseteq V_i^{X_i}$ and $C_i \subseteq V_i^{Y_i}$ with use w, where $C_i := B_i \cup D_i$. Then we remove C_i from T_i^A (by enumerating into A) and restrain $A \upharpoonright w$. Note that since $\mu(V_i^{X_i}) < q_i$ and $\mu(V_i^{Y_i}) < q_i$ the procedure is well defined. Since $X_i \cap Y_i = \emptyset$, either $A \upharpoonright w \neq (X_i \cup Y_i) \upharpoonright w$ or B_i permanently stays in $V_i^{X_i}$ or D_i permanently stays in $V_i^{Y_i}$. If we repeat this procedure $\frac{4}{p_i}$ times then either some round has the first outcome, or one of $T_i^A \not\subseteq V_i^{X_i}$, $T_i^A \not\subseteq V_i^{Y_i}$ holds. In any case R_i is satisfied in a Σ_1^0 way. We say that R_i *requires attention at stage* s if either $u_i[s], B_i[s], D_i[s]$ are undefined, or they are defined and $B_i[s] \subseteq V_i^{X_i}[s]$, $D_i[s] \subseteq V_i^{Y_i}[s]$ and $(X_i[s] \cup Y_i[s]) \upharpoonright w = A[s] \upharpoonright w$ where w is the least number greater than the use of $B_i[s]$ in $V_i^{X_i}$ and $D_i[s]$ in $V_i^{Y_i}$. When we say *the leftmost clopen subset of* Q *of measure* q for a clopen set Q such that $\mu(Q) > q$ we mean the unique subset P of Q which has measure q and the property that if $\beta \in P$ then all reals in Q which are lexicographically smaller than β belong to P.

Construction. At stage s let i be the least number $< s$ such that R_i requires attention at s (if there is no such i, go to the next stage). If $u_i[s] \uparrow, B_i[s] \uparrow, D_i[s] \uparrow$, choose the leftmost clopen set $B_i[s+1] \subseteq 2^\omega - V_i^{X_i}[s]$ of measure $p_i/2$ and the leftmost clopen set $D_i[s+1] \subseteq 2^\omega - V_i^{Y_i}[s]$ of measure $p_i/2$, and put them into T_i^A with *big* use $u_i[s+1]$. Otherwise put $u_i[s]$ into A and set $u_j[s+1] \uparrow, B_j[s+1] \uparrow, D_j[s+1] \uparrow$ for all $j \geq i$.

Verification. By the above discussion the construction is well defined, i.e. when it chooses B_i, D_i, suitable such sets exist. Also note that if $u_i[s] \downarrow, u_j[s] \downarrow$ and $i < j$ then $u_i[s] < u_j[s]$. In particular, as long as $u_i[s] \downarrow$, no requirement R_j with $j > i$ can change $A \upharpoonright (u_i + 1)$. Note that $T_i^A[s] = B_i[s] \cup D_i[s]$ (or $\emptyset$ if $B_i[s], D_i[s] \uparrow$); and if B_i, D_i are set $\uparrow$ at s then A changes below $u_i[s]$. So $T^A = \cup_i T_i^A$ is a $\Sigma_1^0(A)$ class. We have $\mu(T_i^A) < 2^{-i-2}$ since at any time $B_i[t] \cup D_i[t]$ has measure at most p_i, which is $< 2^{-i-2}$.

Next we show by induction that all R_i require attention finitely often and are satisfied. Suppose that this holds for all R_j, with $j < i$ and s_0 is the least stage after all stages where one of these R_j requires attention. At (the beginning of) s_0 we must have $u_i[s_0] \uparrow, B_i[s_0] \uparrow, D_i[s_0] \uparrow$ and so R_i will receive attention at s_0. In the following stages R_i can only redefine its parameters at most $\frac{4}{p_i}$ times, since $\mu(V_i^{X_i}) < q_i$ and $\mu(V_i^{Y_i}) < q_i$. When this stops at some stage s_1, we will have $u_j = u_j[s] \downarrow, B_j = B_j[s] \downarrow, D_j = D_j[s] \downarrow$ for all $s > s_1$ and $j \leq i$, and either $A \upharpoonright w \neq (X_i \cup Y_i) \upharpoonright w$ for some w or $B_i \not\subseteq V_i^{X_i}$ or $D_i \not\subseteq V_i^{Y_i}$, and so R_i is satisfied.

Finally we show that $A \equiv_T \emptyset'$. Let f be a computable function such that $X_{f(i)} = Y_{f(i)} = \emptyset$, $q_{f(i)} = 2^{-1}$ and for all i,

$$V_{f(i)}^{X_{f(i)}} = V_{f(i)}^{Y_{f(i)}} = \begin{cases} \{00\} & \text{if } i \in \emptyset' \\ \emptyset & \text{if } i \notin \emptyset' \end{cases}$$

where '00' is a string representing the leftmost quarter of 2^ω. According to the argument above and the construction, $u_i := \lim_s u_i[s]$ exists for all i and A can compute a modulus of convergence for this function, i.e. there is an A-recursive function Φ^A such that for each i and all $t \geq \Phi^A(i)$ we have $u_i = u_i[t]$. Then according to the construction and since we always choose the leftmost suitable clopen set to enumerate into T^A we have that $i \in \emptyset'$ iff $i \in \emptyset'[\Phi^A(f(i))]$ and so $\emptyset' \leq_T A$. □

Theorem 3.2. If B is c.e. and $B \not\leq_{LR} \emptyset$ then there is a c.e. set $A \leq_T B$ which cannot be split into two c.e. sets X, Y such that $A \equiv_{LR} X \equiv_{LR} Y$.

Proof. We use the ideas and some of the notation in the proof of Theorem 3.1 in a more refined form. Let (U_i) be universal oracle Martin-Löf test and let t_i be the least such that $2^{-t_i} < 2^{-i-2} \cdot (1 - q_i)$ (so that, in particular, $\mu(U_{t_i}^B) < 1 - \mu(V_i))$. Without loss of generality we can assume that

$$\mu(U_{t_i}^B[s]) < 2^{-t_i} < 2^{-i-2} \cdot (1 - q_i) \text{ for all } s. \tag{1}$$

Since $B \not\leq_{LR} \emptyset$ for all Σ_1^0 classes E such that $U_{t_i}^B \subseteq E$ we have $\mu(E) = 1$. To satisfy R_i we will enumerate clopen sets into T_i^A (as before) as well as a Σ_1^0 class E_i such that $U_{t_i}^B \subseteq E_i$. The idea is that, roughly speaking, for any amount that is put into T_i^A (and so $V_i^{X_i}$ or $V_i^{Y_i}$), the same amount is put into E_i. Eventually, the measure in $E_i - U_{t_i}^B$ will translate into measure in $V_i^X - T_i^A$ or $V_i^Y - T_i^A$ (B changes will allow A changes) and by making $\mu(E_i)$ large enough, we have that either $V_i^{X_i}$ or $V_i^{Y_i}$ will stop covering T_i^A; also the measure that permanently stays in T_i^A will be at most $\mu(U_{t_i}^B) < 2^{-i-2}$. Since $\mu(E_i)$ can be at most 1, we will remove $E_i - U_{t_i}^A$ (i.e. the useless measure) from E_i a finite number of times, provided that $\mu(E_i) > 2^{-1} + 2^{-t_i}$. Each time we do this, we ensure that $\mu(V_i^{X_i})$ and $\mu(V_j^{Y_i})$ have increased by a total of 2^{-2}, so that after finitely many times requirement R_i is satisfied.

Order the strings as usual, first by length and then lexicographically. For each stage s and string $\rho \in U_{t_i}^B[s]$ let $p_i(\rho)[s]$ be the stage where ρ was enumerated into $U_{t_i}^B[s]$ with the current computation. At any stage s let $\rho_i[s]$ be the least string in $U_{t_i}^B[s] - E_i[s]$ such that $p_i(\rho_i[s])[s] \leq p_i(\sigma)[s]$ for all $\sigma \in U_{t_i}^B[s] - E_i[s]$ (and $\rho_i[s] \uparrow$ if such string does not exist). Also let $u_i[s]$ be the use of the computation $\rho_i[s] \in U_{t_i}^B[s]$. To *schedule an i-attack at stage s* means to pick a clopen set $C_i^x[s+1] \subseteq 2^\omega - V_i^{X_i}[s]$ of measure $2^{-|\rho_i[s]|-1}$ and a clopen set $C_i^y[s+1] \subseteq 2^\omega - V_i^{Y_i}[s]$ of measure $2^{-|\rho_i[s]|-1}$, and enumerate $C_i[s+1] := C_i^x[s+1] \cup C_i^y[s+1]$ into T_i^A with *big* use $v_i[s+1]$. An attack which was scheduled at stage s is *cancelled* at stage $t > s$ if t is the least stage with $B \restriction u_i[s] \neq B \restriction u_i[t]$. An attack scheduled at stage s is *implemented* at stage $t > s$ if $\rho_i[s]$ is enumerated into E_i at t. If an attack was implemented at stage s and $B[s] \restriction u_i[s] \neq B[t] \restriction u_i[s]$ for some $t > s$ then for the least such stage t we say that *the attack succeeds at stage* t. If an implemented attack succeeds at some stage, we say that it is *successful*; otherwise we say that it is *unsuccessful*. In the following construction when a parameter is not explicitly redefined it retains its previous value and if a string is not explicitly extracted from T_i^A it remains in it (perhaps with a different computation, but then surely with the same A-use). As usual, we assume that $U_i^B[s]$ is prefix-free for all s, as a set of strings.

We say that R_i *requires attention at stage* s if either $v_i[s] \uparrow$, or $v_i[s] \downarrow$, $((X_i \cup Y_i) \restriction v_i)[s] = (A \restriction v_i)[s]$ and one of the following holds:

(I) An i-attack is cancelled at s.

(II) An i-attack was implemented at some stage $t < s$ and it succeeds at stage s.

(III) An i-attack was scheduled at some stage $t < s$, it has not been cancelled or implemented by stage s, $C_i[t] \subseteq V_i^{X_i}[s]$, $C_i[t] \subseteq V_i^{Y_i}[s]$, and $(X_i \cup Y_i)[s] \restriction w = A[s] \restriction w$ for some w greater than the use of $C_i[t]$ in $V_i^{X_i}[s]$ and in $V_i^{Y_i}[s]$.

(IV) All previous attacks have been either implemented or cancelled.

(V) $\mu(E_i[s]) > 2^{-1} + 2^{-t_i}$.

To *initialize* R_i means to empty E_i and T_i^A.

Construction. At stage s pick the least $i < s$ such that R_i requires attention at s (if such i does not exist, go to the next stage) and do the following.

- If an i-attack is cancelled at s, enumerate $v_i[s]$ into A, remove $C_i[s]$ from T_i^A and initialize R_j for all $j > i$.
- If an i-attack was implemented at some stage $t < s$ and it succeeds at stage s, put $v_i[t]$ into A, remove $C_i[t]$ from T_i^A and initialize R_j for all $j > i$.
- If (IV) applies, *schedule an attack* at stage s.
- If (III) applies then enumerate $\rho_i[t]$ into E_i and say that this attack was *implemented* at stage s.
- If $\mu(E_i[s]) > 2^{-1} + 2^{-t_i}$, remove from E_i the set $E_i - U_{t_i}^A[s]$.

Verification. Note that at any stage an attack is scheduled only if all previous attacks are either cancelled or implemented. If an attack is implemented at stage s and another attack is scheduled at $t > s$ (and R_i is not initialized in $[s, t]$) we have $\rho_i[s] | \rho_i[t]$ (so that $[\rho_i[s]] \cap [\rho_i[t]] = \emptyset$). Also, in that case, if $A = X_i \cup Y_i$ we have $C_i^j[s] \cap C_i^j[t+1] = \emptyset$ for some $j \in \{x, y\}$, depending on whether we have $C_i^x[s] \subseteq V_i^{X_i}[t]$ or $C_i^y[s] \subseteq V_i^{Y_i}[t]$ (one of the two must occur since at s an i-attack was implemented and only one of X_i, Y_i may change below the relevant use). In particular, E_i is prefix free and $T_i^A[s]$ is prefix free (as a set of strings) for all s.

By induction on the stages we have that if $i < j$ and $v_j[t]\downarrow$ then $v_i[t]\downarrow$ and $v_i[t] < v_j[t]$. This means that if during initialization the set T_i^A is emptied at stage s, then A changes below the smallest use of existing computations of the form $\sigma \in T_i^A$ for strings σ. So (T_i^A) is indeed a uniform sequence of $\Sigma_1^0(A)$ classes, hence T^A is a $\Sigma_1^0(A)$ class. Moreover by the choice of $v_i[s]$, if $A \restriction n$ changes at stage t then $B \restriction n$ changes at stage t. So $A \leq_{ibT} B$ (where ibT indicates a Turing reduction with the use function being the identity); in particular $A \leq_T B$.

Next we show that each R_i is satisfied and stops requiring attention after some stage. For a contradiction suppose that there is a least i such that either R_i is not satisfied or it requires attention infinitely often. Suppose that s_0 is the least stage such that R_j, $j < i$ do not require attention at any stage $s \geq s_0$. In any case we have $T_i^A \subseteq V_i^{X_i}$ and $T_i^A \subseteq V_i^{Y_i}$ because otherwise some i-attack would never be implemented or cancelled. This means that i-attacks will be scheduled at infinitely many stages (by the choice of t_i and the fact that $\mu(V_i^{X_i}), \mu(V_i^{Y_i})$ are < 1 there will always be a suitable clopen set for scheduling a new attack) and by the definition of $\rho_i[s]$, infinitely many of them will not be cancelled. In fact, if η is a string in $U_{t_i}^B[s]$ with correct B-use, then for some stage t we will have $\rho_i[t] = \eta$ and if t_0 is the least such stage, the attack scheduled at t_0 will be implemented (and will be unsuccessful). This means that if we never removed measure from E_i after stage s_0 (under the fifth condition for R_i to require attention) then $U_{t_i}^B \subseteq E_i$ and since $B >_{LR} \emptyset$ we have $\mu(E_i) = 1$. In particular $\mu(E_i) > 2^{-1} + 2^{-t_i}$ which means that we will remove useless measure from E_i after s_0. The same argument shows that there will be infinitely many stages $s_1, s_2, \dots$ at which we R_i requires and receives attention under the fifth condition. If we let

$$D_i = \{\rho_i[s] \mid \text{at } s \geq s_0 \text{ an unsuccesful } i\text{-attack was implemented}\}$$

then since

$$E_i[s_j] = \{\rho_i[s] \mid \text{at stage } s \in [s_{j-1}, s_j) \text{ an } i\text{-attack was implemented}\}$$

for each $j \in \mathbb{N}$ we have $E_i[s_j] \subseteq D_i \cup H_i[s_j]$, where

$$H_i[s_j] = \{\rho_i[s] \mid \text{at } s \in [s_{j-1}, s_j) \text{ a successful } i\text{-attack was implemented}\}.$$

If at stage s an unsuccessful attack was implemented we must have $\rho_i[s] \in U_{t_i}^B$, so that $D_i \subseteq U_{t_i}^B$ and $\mu(D_i) < 2^{-t_i}$. Since $D_i \cap H_i = \emptyset$ and $\mu(E_i[s_j]) > 2^{-1} + 2^{-t_i}$ we have $\mu(H_i[s_j]) > 2^{-1}$. But every string $\eta \in \cup_j H_i[s_j]$ corresponds to a pair of clopen sets $C_i^x(\eta), C_i^y(\eta)$ such that

- $\mu(C_i^x(\eta)) = \mu(C_i^y(\eta)) = \frac{\mu([\eta])}{2}$
- $C_i^x(\eta)$ stays permanently in $V_i^{X_i}$ or $C_i^y(\eta)$ stays permanently in $V_i^{Y_i}$

- if $\eta_1 = \rho_i[t_1], \eta_2 = \rho_i[t_2]$ for $s_0 \leq t_1 < t_2$ and $C_i^x(\eta_1)$ stays permanently in $V_i^{X_i}$ then $C_i^x(\eta_1) \cap C_i^x(\eta_2) = \emptyset$; if $C_i^y(\eta_1)$ stays permanently in $V_i^{Y_i}$ then $C_i^y(\eta_1) \cap C_i^y(\eta_2) = \emptyset$

which means that at each s_j either $\mu(V_i^{X_i})$ or $\mu(V_i^{Y_i})$ has increased by at least 2^{-2} since s_{j-1}. Since the sequence (s_j) is infinite and $\mu(V_i^{X_i}), \mu(V_i^{Y_i})$ are less than $q_i < 1$, this is a contradiction.

Finally we need to show that $\mu(T^A) < 1$. Let s_0 be as before, and let W be the set of stages $t \geq s_0$ at which an unsuccessful i-attack was implemented. We have $T_i^A = \cup\{C_i[t] \mid t \in W\}$ and $\mu(C_i[t]) = \mu([\rho_i[t]])$ for all $t \in W$. Hence $\mu(T_i^A) = \mu(D_i)$ and since $D_i \subseteq U_{t_i}^B$ we have $\mu(T_i^A) < 2^{-t_i} < 2^{-i-2}$, which shows that $\mu(T^A) < 1$. □

An obvious question which is left unanswered here is whether the c.e. sets of Theorem 3.1 occur in every non-zero LR degree. We conjecture that this is not the case.

Some properties of c.e. LR degrees can be derived from a combination of known properties of the structure of Turing degrees inside an LR degree and properties of the Turing degrees. As an example we demonstrate the following.

Theorem 3.3. Let $n \in \mathbb{N}$. If A is c.e. then there exists B of properly n-c.e. Turing degree such that $A \equiv_{LR} B$.

Proof. Since every c.e. LR degree contains noncomputable c.e. sets, we can assume that A is noncomputable. By a result in [2] we have that there exists a c.e. set C such that $C <_T A$ and $C \equiv_{LR} A$. Then by the density theorem in [6] there is a set B of properly n-c.e. Turing degree such that $C <_T B <_T A$ and so $B \equiv_{LR} A$. □

4. Proof of Theorem 1.1

In the following we fix U to be the second member of a universal oracle Martin-Löf test, so that $\mu(U^X) \leq 2^{-1}$ for all $X \in 2^\omega$. To show Theorem 1.1 it suffices to construct a non-cuppable set A such that $U^{\emptyset'} \subseteq V^A$ for some $\Sigma_1^0(A)$ class V^A of measure < 1.

We adopt the usual assumptions that, for a Turing functional Γ, $\Gamma^X(z)[s]\downarrow$ only if $\Gamma^X(y)[s]\downarrow$ for all $y < z$, and that *use* $\Gamma^X(y)[s] \leq$ *use* $\Gamma^X(z)[s] \leq s$ if $\Gamma^X(z)\downarrow$ and $y \leq z$. A Turing functional Γ may be considered as a c.e. set of axioms $\langle z, y, \sigma\rangle$ (asserting that $\Gamma^X(z) = y$ for all $X \in 2^\omega$ with $\sigma \subset X$), which are consistent in the sense that if $\langle z, y, \sigma\rangle$ and $\langle z, y', \sigma'\rangle$ are both in the set, for $y' \neq y$, then σ and σ' are incomparable. We will abbreviate $\Gamma^{X\oplus Y}$ as Γ^{XY}.

4.1. **Making A non-cuppable.** We describe the basic strategies for a non-cuppable degree, based on [17, 25]. We will construct Turing functionals Δ_e

to ensure that the following holds for all $e \in \omega$:

$$N_e : \qquad \Gamma_e^{AW_e} = K \Rightarrow \Delta_e^{W_e} = \emptyset' \tag{2}$$

where $\langle \Gamma_e, W_e \rangle$ ranges over all pairs of Turing functionals and c.e. sets; assuming that $\emptyset' \subseteq 2\mathbb{N}$ we let $K = D \cup \emptyset'$ where $D \subseteq 2\mathbb{N}+1$ is an auxiliary that we enumerate. In the following discussion we omit the index e. The idea is to let Δ^W copy Γ^{AW} by monitoring the reduction Γ^{AW} and restraining A to preserve the agreement of the two reductions. The problem with this approach is that the restraint on A may well have limit ∞, in which case very little can be done to make A nontrivial, let alone LR-above $\emptyset'$. The solution is to split N into infinitely many subrequirements M_p which are responsible just for the definition of $\Delta^W(p)$, thus splitting an infinite restraint into infinitely many finite restraints. The strategies for the subrequirements M_p will be coordinated by a master N strategy which will make sure that Δ is consistent and this coordination will be implemented on a tree of strategies.

We can think of N having two outcomes $\infty \prec f$ (i.e. ∞ is to the left of f) corresponding to whether there are infinitely many expansionary stages in $\Gamma^{AW} = K$ or not, and M_p outcomes $\infty \prec f$ according to whether $\Gamma^{AW}(p) \uparrow$ or equivalently, $\Delta^W(p) \downarrow$. This induces a uniformly labelled tree of strategies where each level is occupied by either some N or some M_p. For the consistency of Δ we make sure that at any M_p-level (i.e. occupied by an M requirement) and at any stage at most one node α will be responsible for $\Delta^W(p) \downarrow$ (by preserving A in $\Gamma^{AW}(p) \downarrow$). Any nodes to the right of α may adopt that Δ-definition but if a node to the left of α wishes to define $\Delta(p)$ it must first cancel the Δ computation that α holds. This happens by enumerating something into the auxiliary set D which in turn causes a W-change (provided that the Γ reduction is valid). Eventually, if $\Gamma^{AW} = K$, at each M_p level there will be exactly one node *on or to the left of the true path* which permanently preserves $\Delta^W(p) \downarrow = \emptyset'(p)$. Otherwise some node will witness partiality. As in any $\emptyset''$ priority argument the restraints imposed on a node on the true path will be finite.

Each M_p-node α has a *flip-point* d, which is the number enumerated into D when we wish to cancel the computation $\Delta(p) \downarrow$. When α is visited, it checks if the computation $\Gamma^{AW}(d)$ has changed since the last time it was visited and if so, it plays outcome ∞. Otherwise we may define $\Delta^W(p) = \Gamma^{AW}(p)$, with W-use $u = \textit{use}\, \Gamma^{AW}(d)$ and restrain $A \upharpoonright u$. If we later want to visit a node β to the left of α, we enumerate the flip-point d into D whilst maintaining α's A-restraint. This enumeration should force a W-change below u, and so α will not hold a Δ-computation anymore (if this does not happen then N will be satisfied by a finite outcome). Then we can drop the restraint of α and β can take action. This must happen immediately upon seeing the N-expansionary stage, otherwise some other node α' to the right of β may act first and define another Δ-computation which prevents β from being visited. For this reason when we enumerate d into D we create a link

(τ, β) from the N-node τ to β and when τ is next visited at an expansionary stage we will follow the link straight to β.

4.2. **Measure-guessing nodes and LR-completeness.** To make A LR-complete, it suffices to construct a $\Sigma^0_1(A)$ class V^A with $U^{\emptyset'} \subseteq V^A$ and $\mu(V^A) < 1$. Without loss of generality we assume that if $\langle \sigma, \tau \rangle$ is enumerated into U at stage s then $|\sigma| = |\tau| = s$. We will also use the *hat-trick* for $U^{\emptyset'}$: let $k_s = \min\{x : x \in \emptyset'[s] - \emptyset'[s-1]\}$, or $k = s$ if there are no such x and define $\widehat{\emptyset'}[s] = \emptyset'[s] \upharpoonright k$. Then $\widehat{U^{\emptyset'}}[s] = \big\{\sigma : \langle \sigma, \tau \rangle \in U_s \text{ for some } \tau \subseteq \widehat{\emptyset'}[s]\big\}$. In the following we assume that $U^{\emptyset'}[s]$ and $\emptyset'[s]$ refer to $\widehat{U^{\emptyset'}}[s]$ and $\widehat{\emptyset'}[s]$ respectively. Then infinitely often we have *true stages* s at which $U^{\emptyset'}[s] = U^{\emptyset' \upharpoonright n} \subset U^{\emptyset'}$ for some n, and thus $\mu\big(U^{\emptyset'}[s]\big) < \mu\big(U^{\emptyset'}\big)$.

Whenever an interval σ appears in $U^{\emptyset'}$, we add it to V^A with large A-use u. If a $\emptyset'$-change later removes σ from $U^{\emptyset'}$, we could remove it from V^A by enumerating u into A, provided that u is not restrained by some requirement. The A-change may also remove some legitimate intervals from V^A, but we add these again with the same use as before. This clearly gives $U^{\emptyset'} \subseteq V^A$. The main conflict is that the A-restraints will prevent us from removing some superfluous 'junk' intervals σ from V^A. For the argument to succeed, we must ensure that the total measure of junk intervals $\mu\big(V^A - U^{\emptyset'}\big) < \frac{1}{2}$. We assign each requirement (each level of the tree) a quota ϵ, which is the amount of junk measure that requirement is allowed to capture. We implement the negative strategies in such a way that we have at most one node imposing restraint at each level of the tree. A restraint may only be imposed on A if the (current) junk measure that it captures is less than the quota. To ensure that strategies will eventually be able to impose restraints under this restriction, we choose the quota $\epsilon(k)$ of level k of the tree so that $\sum_{j>k} \epsilon(j) < \epsilon(k)$ (in this way the lower priority requirements will not capture more than $\epsilon(k)$ of junk).

To ensure that the strategies do not exceed their junk quota, the predecessor of each N and M node will be a node with a strategy G which measures $\mu(U^{\emptyset'})$ in a Π^0_2 way. The backup nodes G successively subdivide the interval $[0, 1)$, assigning each of its outcomes an interval $[q, r)$ which corresponds to a guess that $\mu\big(U^{\emptyset'}\big) \in [q, r)$. The construction will make sure that if the backing node of a strategy predicts the right interval $[q, r)$ of $\mu\big(U^{\emptyset'}[s]\big)$ then the junk measure that it captures will increase by no more than $r - q$ after it acts. If we choose $r - q = \epsilon$, then α will capture at most 2ϵ of junk, which is acceptable if we choose the quotas $\epsilon(k)$ such that $\sum_{k \in \omega} 2\epsilon(k) < \frac{1}{2}$. An analysis of the permanent restraints and the timing of the enumerations into A in the construction will verify that $\mu\big(V^A - U^{\emptyset'}\big) < \frac{1}{2}$.

4.3. **Combining the strategies.** The difficulty in combining the non-cupping and LR-completeness strategies stems from the fact that the non-cupping subrequirements are not independent of each other or of the parent

N-node. In previous constructions of LR-complete c.e. sets (see [3, 5]) when a node holds a restraint under a measure guess which proves wrong, we initialise that strategy and all lower-priority nodes. However here we can only initialise non-cupping parent N-nodes since by initialising an M-node we may make Δ inconsistent. Once a $\Delta^W(p)$ axiom has been enumerated, we must retain the A-restraint until the axiom is invalidated by a W-change or the parent N-node is initialised.

Thus whenever some M-node holds a restraint under a wrong assumption about $\mu\big(U^{\emptyset'}\big)$ we just try to invalidate the corresponding Δ axiom by enumerating the flip point and waiting for a suitable W-change. The construction will make sure that if this does not happen and N is not reset, the junk measure from the subrequirements of N will be less than the quota of N, even though the junk measure of some M may turn out to be larger than its quota. Overall this satisfies N trivially and with small enough cost. The trick which allows the above quota-junk relation is in enumeration of $U^{\emptyset'}$: it is prefix-free and if some interval σ leaves $U^{\emptyset'}$ then all intervals which were enumerated after σ leave as well, at the same time.

4.4. **Priority Tree and Definitions.** The priority tree is a finite branching tree which consists of the parent nodes labelled N_e, the subrequirement nodes labelled $M_{e,p}$, and the measure-guessing backup nodes labelled G. We adopt the convention that the root node is at the top and the tree branches downwards; thus we may say that a node α is above a node β if α is an ancestor of β. Let $\langle\cdot,\cdot\rangle$ be a monotone 1–1 computable function from $\mathbb{N}\times\mathbb{N}$ onto $\mathbb{N}$. Requirement N_e has code $\langle e,0\rangle$ and $M_{e,2p}$ has code $\langle e,p+1\rangle$ (by assumption $\emptyset'\subset 2\omega$ and so only even $\Delta^W(p)$ arguments need to be considered). We say that requirement R_1 has higher priority than R_2 (writing $R_1 < R_2$) if the code of R_1 is smaller than the one of R_2. We define the tree based on this priority ordering. If $|\alpha| = 2\langle e,0\rangle+1$ then α is labelled N_e and if $|\alpha| = 2\langle e,p+1\rangle+1$ then it is labelled $M_{e,2p}$. If $|\alpha| = 2e$ then α is labelled G.

The N_e-nodes τ have outcomes $\infty \prec f$ and are associated with a functional Δ_τ that is built by the $M_{e,p}$-nodes below τ and is occasionally cleared and started afresh when τ is reset. The $M_{e,p}$ nodes have outcomes $\infty \prec f$ and are associated with a *flip-point* d_α which may change in the course of the construction. A measure-guessing G-node γ has outcomes $q_0 \prec q_1 \prec q_2 \prec q_3$ which correspond to guesses about an interval in which $\mu\big(U^{\emptyset'}\big)$ may lie. Inductively we start with the root node λ, divide $[0,2^{-1})$ (since $\mu\big(U^{\emptyset'}\big)\le\frac{1}{2}$) into four equal intervals and assign them in increasing order to outcomes $q_0 \prec q_1 \prec q_2 \prec q_3$ respectively, which we think of as edges from λ. If $|\gamma| = 2e$ and is below interval-outcome I of $\gamma \restriction 2e-2$, divide I into four equal intervals and assign them in increasing order to outcomes $q_0 \prec q_1 \prec q_2 \prec q_3$ respectively, which we think of as edges from γ.

For an M or N-node α, with $\alpha = \gamma^\frown x$ for a G-node γ, let $I_\alpha = [q,r)$ be the interval assigned to outcome x of γ. We write $q(\alpha)$ for the lower endpoint q of I_α, and $\epsilon(\alpha)$ for $r-q$, the width of I_α. We refer to $\epsilon(\alpha)$ as α's *resolution* and $q(\alpha)$ as its *measure guess.* Since all nodes of the same label have the same length, we may write $\epsilon(N_e)$ or $\epsilon(M_{e,p})$ to denote $\epsilon(\alpha)$ for any node α labelled N_e or $M_{e,p}$, respectively. For each N or M requirement R we have

$$\sum_{R'>R} 2\epsilon(R') < \epsilon(R) \quad \text{and} \quad \sum_{e\in\omega} 2\epsilon(N_e) < \frac{1}{2} \tag{3}$$

where R' is an N or M requirement. The ordering $\prec$ on the outcomes is extended to the nodes of the tree lexicographically: $\alpha \prec \beta$ if for the longest common initial segment γ of those nodes, $\gamma^\frown x \subseteq \alpha$ and $\gamma^\frown y \subseteq \beta$ for $x \prec y$. We say that α has *higher priority than* β if either $\alpha \subset \beta$ or $\alpha \prec \beta$. We write r_α for the restraint imposed on A by node α, and α^- for the predecessor of α. Also let $R_\alpha = \max\{r_\beta : \beta \prec \alpha \text{ or } \beta \subset \alpha\}$. All parameters have a current value each time they are mentioned in the construction and their value at the beginning of stage s is indicated by the suffix $[s]$. For an $M_{e,p}$-node α, we write $\tau(\alpha)$ for the unique N_e-node $\tau \subset \alpha$. We refer to τ as α's *parent*, or say that α is *working for* τ. An $M_{e,p}$-node α with parent τ is *enabled* if $\tau^\frown\infty \subset \alpha$ and for every $M_{e,p'}$-node α' with $\tau \subset \alpha' \subset \alpha$, we have $\alpha'^\frown f \subset \alpha$. Otherwise, α is *disabled* (which means that it regards Γ^{AW_e} as partial and no further action is needed for N_e).

4.5. **Construction.** Set $A[0] = \emptyset, \Delta_\tau = \emptyset$ for all N-nodes τ, and $d_\alpha \uparrow, r_\alpha = 0$ for all M-nodes α. When a parameter is assigned a value, it retains that value until explicitly given a new value. To *reset* an N-node τ means to empty Δ_τ, set $r_\beta = 0$ and $d_\beta \uparrow$ for any M-nodes β working for τ, and remove any links to or from τ or any M-node β working for τ. To *reset* an M-node α means to remove any links to it and if $r_\alpha \neq 0$ and $d_\alpha \downarrow$, enumerate d_α into D, setting $d_\alpha \uparrow$. To *reset* a G-node means to remove any links to it. The construction will explicitly declare certain nodes α to be *accessible* at each stage, which does not merely mean that $\alpha \subset \delta_s$. If α is an N-node, it will also declare certain stages to be α*-expansionary.* We give the enumeration of V^A during the stages s of the construction in advance:

(4) **Enumeration of V^A.** For each $\langle \sigma, \rho \rangle \in U[s]$ with $\rho \subset \emptyset'[s]$ but $\sigma \notin V^A[s]$, if $\sigma \in V^A[t]$ with use u for some $t < s$ take the largest such t and if $\langle \sigma, \rho' \rangle \in U[t]$, $\rho' \subset \emptyset'[s]$, then enumerate σ into $V^A[s+1]$ with use u. Otherwise, put σ into $V^A[s+1]$ with fresh use.

The construction will occasionally call the following routine, which is needed in order to access certain outcomes x of nodes α.

(5) **Routine** $L(\alpha, x, s)$**.** Reset all N-nodes which are on the left of $\alpha^\frown x$. Then consider the longest node $\tau \subset \alpha$ which has label N_e for some $e \in \mathbb{N}$ and there is some $M_{e,p}$-node $\beta \supset \tau$ with $\beta \succ \alpha^\frown x$, $r_\beta[s] \neq 0$. If τ exists let β be the shortest node as above, enumerate d_β into D (if $d_\beta \downarrow$), set $d_\beta \uparrow$, create a link (τ, α) associated with outcome x and go to step 4. Otherwise let $\delta_{s,t+1} = \alpha^\frown x$ and go to step 3.

At stage s, we perform the following steps in order.

Step 1. *(Reset some nodes)* Look for the highest priority node α such that some $\beta \supseteq \alpha$ has been accessed since α was last reset and $\mu\big(U^{\emptyset'}[s]\big) < q(\alpha)$. If there is such, reset α and all nodes of lower priority than α.

Step 2. *(Drop some restraints)* For each M-node α with $r_\alpha \neq 0$ and $W \restriction r_\alpha[s] \neq W \restriction r_\alpha[t]$, where t is the stage for which the restraint r_α was last set, set $r_\alpha = 0$ and reset $\alpha^\frown f$ and all nodes of lower priority than $\alpha^\frown f$.

Step 3. *(Define δ_s in substages)* Let $\delta_{s,0} = \lambda$. Let t be the largest number such that $\delta_{s,t} \downarrow$. If $|\delta_{s,t}| \geq s$ then go to step 4. Otherwise let $\alpha = \delta_{s,t}$ and check if

$$\text{(6)} \quad \text{there is an } M\text{-node } \beta \preceq \alpha \text{ with } \tau(\beta)^\frown\infty \subset \alpha, r_\beta \neq 0 \text{ and } d_\beta \uparrow .$$

If so, go to step 4; otherwise declare α accessible and go to the relevant clause below.

- α **is a** G**-node**. Let $[a_0, a_1), \dots [a_3, a_4)$ be the intervals corresponding to the outcomes of α and $\epsilon = a_1 - a_0$ be the resolution of α. Let $g_\alpha(s)$ be the largest $t < s$ such that $\alpha \subset \delta_t$, or 0 if such t does not exist. Let

$$\text{(7)} \quad \nu = \nu(\alpha, s) = \min\big\{\mu\big(U^{\emptyset'}[k]\big) : g_\alpha(s) < k \leq s \;\&\; \mu\big(U^{\emptyset'}[k]\big) \in [a_0, a_4)\big\}$$

(Lemma 4.2 verifies that ν always exists) and let i be such that $\nu \in [a_i, a_{i+1})$, and run routine $L(\alpha, q_i, s)$.

- α **is an** $M_{e,p}$**-node**. If it is a disabled $M_{e,p}$-node, let $\delta_{s,t+1} = \alpha^\frown\infty$ and go to step 3. Otherwise do as follows. Let $d = d_\alpha, \tau = \tau(\alpha), W = W_e, \Gamma = \Gamma_e$, $u =$ *use* $\Gamma^{AW}(d)[s]$ (if defined) and

$$\text{(8)} \qquad h_\alpha(s) = \max\{t \leq s : \nu(\alpha^-, s) = \mu\big(U^{\emptyset'}[t]\big)\}$$

where α^- is the predecessor of α. $h_\alpha(s)$ *is the stage for which the measure-guessing* G*-node of* α *gave its outcome.* If $d \uparrow$ choose a fresh value for d.

M1. If $\Delta^W_\tau(p)[s] \downarrow$ let $\delta_{s,t+1} = \alpha^\frown f$ and go to step 3; if $\Delta^W_{\tau'}(p)[s] \downarrow$ for some N_e-node $\tau' \prec \alpha$ then define $\Delta^W_\tau(p) = \Delta^W_{\tau'}(p)$ with the same use, let $\delta_{s,t+1} = \alpha^\frown f$ and go to step 3.

M2. Otherwise if $\Gamma^{AW}(d)[s] \uparrow$ or if $A \restriction u[s] \neq A \restriction u[t]$ or $W \restriction u[s] \neq W \restriction u[t]$ for the last stage t when α was accessible, or if α has never been accessible before, then run routine $L(\alpha, \infty, s)$.

M3. Otherwise, if

$$\mu\Big(V^{A\restriction u}[s] - V^{A\restriction R_\alpha}[s] - U^{\emptyset'}[h_\alpha(s)]\Big) < \epsilon(\alpha) \tag{9}$$

we define $\Delta^W_\tau(p) = \Gamma^{AW}(p)[s]$ with use u, impose restraint $r_\alpha[s+1] = u$, and go to step 4.

M4. In any other case go to step 4.

- α **is an** N_e**-node**. Let $l(\alpha, s) = \min\{n : \Gamma_e^{AW_e}(n)[s] \neq K(n)[s]\} \cup \{d : d$ was enumerated into D in step 1 or 2$\}$, and say that stage s is α-*expansionary* if $l(\alpha, s) > l(\beta, t)$ for all N_e-nodes $\beta \preceq \alpha$ and all $t < s$ such that β was accessible at t. If s is not α-expansionary, then let $\delta_{s,t+1} = \alpha^\frown f$ and go to step 3. Otherwise, if there is a link (α, β) associated with outcome x of β which was created at stage $t < s$, remove it and run routine $L(\beta, x, s)$. Otherwise run routine $L(\alpha, \infty, s)$.

Step 4. Set $\delta_s = \alpha$ for the longest α which was declared accessible in step 3. Reset all nodes $\succ \delta_s$ and enumerate into A the least number which is not in A and is greater than all $r_\beta[s+1]$ for all M-nodes β.

4.6. **Verification.** In the following, whenever we say 'M-node' we mean an enabled M-node, as disabled M-nodes have no effect on the construction. A basic fact which stems from the the hat-trick in the enumeration of $U^{\emptyset'}$ and will be used repeatedly in the verification is the following: if $s_0 < t \leq s_1$ are stages and $\mu(U^{\emptyset'})$ takes its minimum value in $(s_0, s_1]$ at t, then $U^{\emptyset'}[t] \subseteq U^{\emptyset'}[s]$ for all $s \in (s_0, s_1]$.

Lemma 4.1. Links can never be nested or crossing. That is, if (τ, α) and (τ', α') are two distinct links both present at stage s, with $\tau \subset \alpha \subset \beta$ and $\tau' \subset \alpha' \subset \beta$ for some node β, then $\alpha \subset \tau'$ or $\alpha' \subset \tau$. Furthermore, at the end of any stage s, there is at most one link (τ, α) with $\tau \subset \alpha \subseteq \delta_s$, and such a link was created at stage s.

Proof. By induction on the stages. Note that initially there are no links and at any stage at most one link is created. Suppose that the claim holds at stage s and a link (τ, α) is created at stage $s+1$. Then α is accessible at stage $s+1$ or a link was travelled to α, and any links (τ', α') with $\tau' \subset \alpha' \subseteq \alpha$ present at the start of stage $s+1$ have been travelled and removed. If there was a link (τ'', α'') at the start of stage $s+1$ for some $\tau'' \subset \alpha \subset \alpha''$, then that link would have been travelled and α would not be accessible. Thus the new link cannot be crossing or nested within an existing link. Finally any links (τ, α) with $\tau \subset \alpha \subset \delta_{s+1}$ which are present at the start of stage $s+1$, would be travelled and removed during the definition of δ_{s+1} in step 3. Since at most one link is created under routine (5), the last claim of the lemma holds. □

For a G-node γ, let $I_\gamma = [a_0, a_4)$ be the interval being subdivided by γ. The following lemma verifies that a G-node will always have a valid outcome to play when it is accessible.

Lemma 4.2. Suppose a G-node γ is accessible at stage s_0 and let $s_1 = g_\gamma(s_0)$ be the greatest stage $< s_0$ such that $\gamma \subset \delta_{s_1}$ (or 0 if such stage does not exist). Then there is some t with $s_1 < t \leq s_0$ and $\mu\big(U^{\emptyset'}[t]\big) \in I_\gamma$. Thus, when γ is accessible in step 3, ν (as in (7)) will exist.

Proof. Let γ, s_0 and s_1 be as in the lemma. The proof is by simultaneous induction on the length of γ and the stage s_0. For the root node the claim is trivial, so let $|\gamma| > 1$ and suppose that the claim is true for all G-nodes shorter than γ and at all stages $\leq s_0$. Let $\gamma' = \gamma \restriction |\gamma| - 2$ be the last G-node above γ and note that if γ has never been accessed before, a suitable t must exist or else γ' would not have chosen the outcome leading to γ. Suppose then that γ has been accessed before. If γ' is also accessible at s_0, since $\gamma' \subset \gamma$ we have $g_{\gamma'}(s_0) \geq s_1$ and by hypothesis there is a suitable t with $g_{\gamma'}(s_0) < t \leq s_0$ and $\mu\big(U^{\emptyset'}[t]\big) \in I_\gamma$.

If γ' is not accessible at s_0, then there must be a link (τ, β) at s_0, with $\tau \subset \gamma' \subseteq \beta \subset \gamma$. Also by induction hypothesis there must be a stage $t_0 < s_0$ such that γ' is accessible at t_0 and $\mu\big(U^{\emptyset'}[t]\big) \in I_\gamma$ for some t with $g_{\gamma'}(t_0) < t \leq t_0$. We can assume that t_0 is the greatest stage $< s_0$ with the above property. If t_2 is the stage at which the link (τ, β) was created we have $t_2 \geq t_0$. Now $\delta_s \not\supseteq \gamma$ for $t_0 \leq s \leq t_2$, as otherwise t_0 would not be the greatest with the above property. Also $\delta_s \not\supseteq \gamma$ for $t_2 < s < s_0$ as otherwise the link would be travelled and removed before s_0, because by Lemma 4.1 links cannot be nested. So $s_1 < t_0$ and $s_1 \leq g_{\gamma'}(t_0)$ since $\gamma' \subset \gamma$, which means that $s_1 < t \leq s_0$. $\square$

By the construction, if an $M_{e,p}$-node α has $r_\alpha[s] \neq 0$ and $d_\alpha\downarrow$, then d_α has not been enumerated into D via resetting or routine (5). Conversely, $r_\alpha[s] \neq 0$ and $d_\alpha\uparrow$ indicates that the construction has attempted to invalidate α's $\Delta^W(p)$ computation. The definition of τ-expansionary stage and the check for (6) in step 3 ensures that no M_e-node of lower priority than α will be accessible again until the $\Delta^W(p)$ computation is invalidated.

A restraint r_α is called *permanent at stage s* if $r_\alpha[s] = r_\alpha[t] \neq 0$ for all $t \geq s$; it is called *permanent* if it is permanent at some stage. Let P be the set of nodes with permanent restraints.

For an M-node α, let $J_\alpha[s] = \{\sigma \in V^A[s+1] - U^{\emptyset'}[s] : R_\alpha[s+1] \leq \mathit{use}\,\sigma < r_\alpha[s+1]\}$, which is the junk intervals that are restrained at stage s by α but not by any higher-priority node at the *end* of stage s. For an N_e-node τ, let $Q_\tau[s] = \bigcup J_\alpha[s]$, where the union is taken over all M_e-nodes α which are either $\supset \tau$ or $\prec \tau$. The following lemma shows that if the junk captured by an M-node becomes greater than the node's quota 2ϵ then the node is reset; and although an M-node may sometimes capture more than its quota of junk (if the junk is never released via step 2), the total junk captured by nodes belonging to an N-node remains within the N-node's quota.

Lemma 4.3. Let β be an M-node and s a stage such that $r_\beta[s+1] \neq 0$ and $d_\beta[s+1]\downarrow$ (so β has not been reset since r_β was set $\neq 0$). Then $\mu(J_\beta[s]) < 2\epsilon(\beta)$. Let τ be an N-node. Then $\mu(Q_\tau[s]) < 2\epsilon(\tau)$ for all s.

Proof. Suppose β and s are as in the first claim. Let t be the stage when $r_\beta[s+1]$ was set. At t, $V^{A\restriction r}[t] = V^{A\restriction r}[s+1]$ for $r = r_\beta[s+1]$ as new intervals in V^A have use chosen fresh. So,

$$\begin{aligned} \mu(J_\beta[s]) &= \mu\big(V^{A\restriction r_\beta}[s+1] - V^{A\restriction R_\beta}[s+1] - U^{\emptyset'}[s]\big) \\ &\leq \mu\big(V^{A\restriction r_\beta}[t] - V^{A\restriction R_\beta}[t] - U^{\emptyset'}[h_\beta(t)]\big) \\ &\quad + \mu\big(U^{\emptyset'}[h_\beta(t)] - U^{\emptyset'}[s]\big) \end{aligned} \tag{10}$$

where the first term of (10) is the junk that β captured when it imposed its restraint $r_\beta[s+1]$, and the second is the measure which appears to be in $U^{\emptyset'}$ at $h_\beta(t)$ but later is removed from $U^{\emptyset'}$. By (9) the first term is less than $\epsilon(\beta)$. Suppose that $\mu\big(U^{\emptyset'}[h_\beta(t)] - U^{\emptyset'}[s]\big) \geq \epsilon(\beta)$. We have $U^{\emptyset'}[h_\beta(t)] - U^{\emptyset'}[t] = \emptyset$, as otherwise (by the canonical enumeration of $U^{\emptyset'}$) there would be a stage $t', h_\beta(t) < t' \leq t$ with $\mu(U^{\emptyset'}[t']) < \mu(U^{\emptyset'}[h_\beta(t)])$, which contradicts (8). So we must have $\mu\big(U^{\emptyset'}[t] - U^{\emptyset'}[s]\big) \geq \epsilon(\beta)$. But then, again by the canonical enumeration of $U^{\emptyset'}$ there would be a stage $t', t < t' \leq s$ such that $\mu(U^{\emptyset'}[t']) \leq \mu(U^{\emptyset'}[h_\beta(t)]) - \epsilon(\beta)$, and β would be reset at t' by step 1 of the construction. So $\mu(U^{\emptyset'}[h_\beta(t)] - U^{\emptyset'}[s]) < \epsilon(\beta)$, and $\mu(J_\beta[s]) < 2\epsilon(\beta)$.

Next, let τ be an N_e-node; we need only consider the case where there is some M_e-node $\beta \supset \tau$ with $J_\beta[s] \neq \emptyset$. Let Z denote the set of M_e-nodes $\beta' \supset \tau$ or $\prec \tau$ with $r_{\beta'}[s+1] \neq 0$, and let β be the longest; by assumption $\beta \supset \tau$. Let t be the stage when $r_\beta[s+1]$ was set $\neq 0$. At t, $d_{\beta'}[t+1]\downarrow$ for all $\beta' \in Z$, as otherwise β would not be accessible at t. Also $\mu(J_\beta[t]) < \epsilon(\beta)$ by (9). So by the first part of the lemma and (3), $\mu(Q_\tau[t]) < 2\epsilon(\tau)$. Also, $d_{\beta'}[t'+1]\downarrow$ for all $t < t' \leq s$ and $\beta' \in Z, \beta' \prec \tau$, as otherwise τ would be reset, contradicting the definition of t. So if $\mu(Q_\tau[t']) \geq 2\epsilon(\tau)$ at some $t < t' \leq s$ it must be because $\sum_{\tau \subset \beta' \in Z} \mu(J_{\beta'}[t']) > \epsilon(\tau)$. But then by the canonical enumeration of $U^{\emptyset'}$ there would be a stage t'' such that $t < t'' \leq t'$ and $\mu(U^{\emptyset'}[t'']) < \mu(U^{\emptyset'}[h_\beta(t)]) - \epsilon(\tau)$. In such a case τ would be reset at step 1, again contradicting the definition of t. So $\mu(Q_\tau[s]) < 2\epsilon(\tau)$. □

In the following lemma we prove simultaneously that the true path $TP = \liminf_s \delta_s$ is infinite, that every node on it has infinitely many chances to act, and that eventually the measure condition (9) will be satisfied for each M-node on TP.

Lemma 4.4. If α is the leftmost node of length $|\alpha|$ such that $\alpha \subseteq \delta_s$ for infinitely many s, then

(1) α is reset only finitely often; if it is an M-node then eventually the flip-point d_α is fixed;
(2) α is accessible infinitely often;

(3) there is some extension $\beta \supset \alpha$ with $\beta \subseteq \delta_s$ for infinitely many s.

Thus $TP = \liminf_s \delta_s$ is infinite.

Proof. First of all, if $|\alpha| = 0$ then $\alpha \subseteq \delta_s$ for all s so 1-3 of the lemma implies that TP is infinite. Then it remains to assume that α is the leftmost node of length $|\alpha|$ such that $\alpha \subseteq \delta_s$ infinitely often and (inductively) that the lemma holds for all $\beta \subset \alpha$, and show claims 1-3.

For the first claim note that there are four places in the construction where α may be reset: in step 1, step 2, step 3 (through the routine L) and step 4. Let s_0 be the second stage such that $\alpha \subseteq \delta_{s_0}$, $\delta_s \not\prec \alpha$ $\forall s > s_0$, any computations $\Delta^W_{\tau(\beta)}(p)\downarrow$ of nodes $\beta \prec \alpha$ that exist at s_0 are permanent and no nodes above or to the left of α are reset after s_0. After s_0, α will not be reset in step 4. If α was reset after s_0 at step 3 then it would be because routine $L(\beta, x, s)$ was run for some $\beta \subset \alpha$ such that $\beta^\frown x \prec \alpha$. But this would mean that either $\delta_s \prec \alpha$ for some $s > s_0$ or α is not $\subseteq \delta_s$ infinitely often, a contradiction.

If α was reset by step 2, by the choice of s_0 there must be some M-node β such that $\beta^\frown f \subset \alpha$ which had a computation $\Delta^W_{\tau(\beta)}(p)\downarrow$ and this was spoilt after s_0. But then the corresponding Γ computation (which has larger use) would be spoilt and the construction would define δ_s to the left of α at M2, a contradiction. Suppose that α was reset in step 1 after stage s_0. By the choice of s_0 there must be a node $\beta \subset \alpha$ and a stage $s_1 > s_0$ such that $\mu\big(U^{\emptyset'}[s_1]\big) < q(\beta)$. But in that case after stage s_1 the construction would define δ_s to the left of α, before it defines it below α, a contradiction. Finally suppose that α is an M-node and d_α was changed after stage s_0. Since α is not reset after s_0 there must be some $\beta \subset \alpha$ which ran routine $L(\beta, x, s_1)$ for $s_1 > s_0$ and $\beta^\frown x \prec \alpha$. But in that case the construction would define δ_s to the left of α, before it defines it below α, a contradiction.

For claim 2, notice that since by hypothesis $\alpha \subseteq \delta_s$ for infinitely many s, the only way that α may stop being accessible after some stage is that for all sufficiently large stages there is a link (τ, β) with $\tau \subset \alpha \subset \beta$. Suppose, for a contradiction, that this is the case and after stage s_0 α is never accessible again. Let $Y[s]$ be the finite set of Δ-computations that are held by M-nodes below α at $s \geq s_0$. Note that if $\delta_t \supseteq \alpha$ for $t \geq s_0$ then by Lemma 4.1 a link must be created at t as otherwise the next time $\alpha \subseteq \delta_s$, α would not be covered by a link and would be accessible. Thus no new computations can be added to Y after s_0 as if a Δ-definition is made then no link is created at that stage. Also, by the construction there are no Δ-computations held by nodes $\succ \alpha$ at the end of a stage s when $\alpha \subseteq \delta_s$. Finally a link is only travelled if the Δ-computation for which it was created has been invalidated. So any link covering α at $s \geq s_0$ is created because of a computation in Y, which is removed from Y when the link is travelled. Since Y is finite and non-increasing, after finitely many stages Y will be empty and α will be accessible when next $\delta_s \supseteq \alpha$.

For claim 3, since α is accessible infinitely often the only way the claim could fail is if, whenever α is accessible after some finite stage $s_0 > |\alpha|$, step 3 is ended without any $\alpha^\frown x$ being declared accessible. Suppose this is the case. Then whenever α is accessible after s_0, step 3 is ended by routine L, or by M3 or M4 if α is an M-node, or because of (6).

At s_0 there are only finitely many $\Delta(p)$ definitions held by nodes β below α. If (6) holds at $s > s_0$ for some $\alpha^\frown x$, it is because one such β was reset while $\tau(\beta)$ was covered by a link. But the link is removed after being travelled, and the next time $\tau(\beta)^\frown\infty \subset \alpha$ is accessible, β's $\Delta(p)$ definition will have been set to 0 at step 2. Since no β below α is accessible after s_0, this can happen only finitely often for the finitely many $\Delta(p)$ computations below α. So it will not happen after some stage s_1.

If step 3 is ended after s_1 due to a routine $L(\alpha, x, s)$ for some outcome x of α, according to the induction hypothesis for α the routine will eventually define $\delta_{s,t} = \alpha^\frown x$ and so $\delta_s \supseteq \alpha^\frown x$ at some stage s. If step 3 is ended because of $M3$ applied to α, then either the Δ-definition made there is permanent (in which case $\alpha^\frown f \subseteq \delta_s$ at some later stage s) or it is not, in which case routine $L(\alpha, \infty, s)$ will be called and the previous argument applies.

Finally, suppose that whenever an $M_{e,p}$-node α is accessible after some s_1, case M4 applies and step 3 is ended at α. We show that eventually the measure condition (9) is satisfied and M3 will apply, a contradiction. At s_1, there are only finitely many nodes $\supset \alpha$ with restraints, and no nodes below α are accessible after s_1. Let s_2 be the second stage after s_1 such that

- any non-permanent restraints below α have been dropped;
- all nodes β above or left of α have settled; ie β is not reset after s_2 and if $r_\beta[s_2] \neq 0$ then $r_\beta[s_2]$ is permanent;
- $\Gamma^{AW}(d_\alpha)\downarrow$ and the use is correct;
- $V^{A\restriction u}[s_2] - V^{A\restriction R_\alpha}[s_2] - U^{\emptyset'}[s_2] = V^{A\restriction u} - V^{A\restriction R_\alpha} - U^{\emptyset'}$;
- α is accessible at s_2.

Such stage exists by the induction hypothesis and the fact that new intervals in V^A have use chosen fresh. Every interval in $V^{A\restriction u}[s_2] - V^{A\restriction R_\alpha}[s_2] - U^{\emptyset'}[s_2]$ is in $J_\beta[s_2]$ for some $\beta \supset \alpha$, as otherwise it would be removed in step 4 contradicting the choice of s_2. Letting $E = \{\beta : \beta \supset \alpha \text{ and } r_\beta[s_2] \neq 0\}$, we have

$$\mu\big(V^{A\restriction u}[s_2] - V^{A\restriction R_\alpha}[s_2] - U^{\emptyset'}[s_2]\big) = \sum_{\beta \in E} \mu(J_\beta[s_2]).$$

Write $E = F \cup G$ where

$$F = \{\beta \in E : \tau(\beta) \subset \alpha\};\;\; G = \{\beta \in E : \alpha \subset \tau(\beta)\}.$$

Note that at s_2, every node β in F has $d_\beta[s_2+1]\downarrow$; as otherwise β has been reset at some $t, s_0 \leq t \leq s_2$, and by choice of s_2 r_β is never set to 0 and β's Δ-definition is never invalidated. But then $\tau(\beta)$ has only finitely many expansionary stages, contradicting that $\tau(\beta)^\frown\infty \subset \alpha$ is accessible infinitely often by induction hypothesis.

Observe that the first clause of Lemma 4.3 holds for any $\beta \in F$ and $s = s_2$, and the second for $\tau = \tau(\beta)$ for any $\beta \in G$ and $s = s_2$. So by (3),

$$\begin{aligned}\mu\big(V^{A\restriction u}[s_2] - V^{A\restriction R_\alpha}[s_2] - U^{\emptyset'}[s_2]\big) &= \sum_{\beta\in F}\mu(J_\beta[s_2]) + \sum_{\tau\in\{\tau(\beta):\beta\in G\}}\mu(Q_\tau[s_2]) \\ &< \sum_{\beta\in F}2\epsilon(\beta) + \sum_{\tau\in\{\tau(\beta):\beta\in G\}}2\epsilon(\tau) \\ &< \epsilon(\alpha).\end{aligned}$$

Thus (9) will hold at s_2, α will make a $\Delta(p)$ definition which will be permanent, and $\alpha^\frown f$ will be accessible at some stage after s_2. □

Lemma 4.5. All non-cupping requirements N_e are satisfied.

Proof. Let τ be the N_e-node on TP. It is clear from the construction that $\tau^\frown\infty \subset TP$ iff there are infinitely many τ-expansionary stages. By Lemma 4.4 and the construction, if α is an M_e-node with $\tau^\frown\infty \subset \alpha \subset TP$ then

- $\alpha^\frown\infty \subset TP \Rightarrow \Gamma^{AW}(d_\alpha)\uparrow$, and
- $\alpha^\frown f \subset TP \Rightarrow \Delta_\tau^{W_e}(p)\downarrow$.

To show that for each e the requirement N_e is satisfied assume that $\Gamma_e^{AW_e} = K$ and let τ be the N_e-node on TP. Since $\Gamma_e^{AW_e} = K$ there are infinitely many τ-expansionary stages. First note that by the construction, Δ_τ is consistent, i.e. at each stage s if $\langle\sigma, n, x\rangle, \langle\rho, n, y\rangle \in \Delta_\tau[s]$ and $\sigma \subseteq \rho$ then $x = y$. Also by Lemma 4.4 and the fact that all strategies appear along the true path, the function Δ_τ^W is total and the restraints imposed by each M_e-node below τ when it makes a definition ensure that $\Delta_\tau^W(p) = \Gamma_e^{AW_e}(p) = \emptyset'(p)$ for each $p \in \mathbb{N}$. Thus $W \geq_T \emptyset'$ and N_e is satisfied. □

Lemma 4.6. $\emptyset' \leq_{LR} A$.

Proof. We must verify that $U^{\emptyset'} \subseteq V^A$ and $\mu\big(V^A\big) < 1$. Once an interval σ appears in $U^{\emptyset'}$ with correct $\emptyset'$-use, according to (4) in any later stage it will be in V^A with the same A-use. Thus eventually it will permanently belong to V^A and $U^{\emptyset'} \subseteq V^A$.

To verify $\mu(V^A) < 1$, since $\mu(U^{\emptyset'}) < \frac{1}{2}$ it suffices to show that $\mu(V^{A\restriction n}[s] - U^{\emptyset'}[s]) < \frac{1}{2}$ for all $n \in \mathbb{N}$ and all $s \geq$ some s_0. Fix n and let s_0 be a stage such that $A \restriction n[s_0] = A \restriction n$ and $V^{A\restriction n}[s_0] - U^{\emptyset'}[s_0] = V^{A\restriction n} - U^{\emptyset'}$. Then for all $s \geq s_0$ we have

$$V^{A\restriction n}[s] - U^{\emptyset'}[s] \subseteq \bigcup_{\tau\subset\delta} Q_\tau[s]$$

where τ runs over the N-nodes and δ is the rightmost path of the tree. Hence, by Lemma 4.3 and the second clause of (3) we have, for $s \geq s_0$,

$$\mu\big(V^{A\restriction n}[s] - U^{\emptyset'}[s]\big) \leq \sum_e 2\epsilon(N_e) < \frac{1}{2}.$$

□

This concludes the proof of Theorem 1.1.

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