Almost-Everywhere Domination, Non-cupping and LR-reducibility

George Barmpalias and Anthony Morphett
University of Leeds

22 June 2007

CiE 2007, Siena

Main result

Theorem

There is a non-cuppable, almost-everywhere dominating c.e. set A.

Definitions

Turing reducibility: $A \leq_T B$ if some oracle Turing machine computes A when given oracle B: $\Gamma^A = B$

Turing degree: equivalence class under $\equiv_T$: $A \leq_T B$ and $B \leq_T A$
$\mathbf{a} = \deg A =$ Turing degree of A

c.e. Turing degrees: those which contain a c.e. set

Join: alternate the bits of A, B

$$A \oplus B = a_0 b_0 a_1 b_1 \ldots$$

Gives *least upper bound* in T-degrees: $\mathbf{a} \cup \mathbf{b} = \deg A \oplus B$

Definitions

$0'$: Halting problem

A': Halting problem relative to A (*Jump* of A)

2^ω: Cantor space of infinite binary strings

$\mu(V)$: Lebesgue measure of $V \subseteq 2^\omega$

Definitions

- f *dominates* g if $f(n) \geq g(n)$ for all but finitely many n.

- A is *almost-everywhere dominating* if there is a total function $f \leq_T A$ such that

$$\mu\Big(\{X \in 2^\omega : f \text{ dominates all total functions } g \leq_T X\}\Big) = 1$$

- A is *non-cuppable* if $\nexists$ a c.e. set $W <_T \emptyset'$ such that

$$A \oplus W \equiv_T \emptyset'.$$

That is, if $A \oplus W \geq_T \emptyset'$, then $W \geq \emptyset'$.
In terms of degrees, $\mathbf{a} \cup \mathbf{w} = \mathbf{0'} \Rightarrow \mathbf{w} = \mathbf{0'}$

Definitions

Lowness and Highness:

A is *low* if $A' \equiv_T \emptyset'$

- jump is as low as possible

A is *high* if $A' \equiv_T \emptyset''$

- jump is as high as possible

Almost Everywhere Domination

Domination suggests highness...

How high are AED sets?

- They are high: B AED $\Rightarrow B' \equiv_T \emptyset''$
- But can be lower than $\emptyset'$: AED $B <_T \emptyset'$ constructed by Cholak, Greenberg, Miller

Non-cupping

NCup $= \{$non-cuppable c.e. degrees$\}$

- First studied by Yates, Cooper $\sim$ 1972
- Harrington (D. Miller), 1970's and 80's
- More recently by Li, Slaman & Yang; Yu & Yang; tree construction

Non-cupping

NCup $= \{$non-cuppable c.e. degrees$\}$

• First studied by Yates, Cooper $\sim$ 1972

• Harrington (D. Miller), 1970's and 80's

• More recently by Li, Slaman & Yang; Yu & Yang; tree construction

NCup forms an ideal:

- closed under $\oplus$
- closed downwards under $\leq_T$

Theorem (Cooper, Yates)

There is a nontrivial non-cuppable c.e. degree.

Theorem (Harrington)

1. *There is a high non-cuppable c.e. degree.*
2. *Moreover, for any high* $\mathbf{b}$ *there is a high* $\mathbf{a}$ *such that* $\mathbf{a}$ *cannot be cupped to* $\mathbf{b}$*:*

$$\forall \mathbf{x}\ \ \mathbf{a} \cup \mathbf{x} \geq \mathbf{b} \Rightarrow \mathbf{x} \geq \mathbf{b}.$$

Theorem (Cooper, Yates)

There is a nontrivial non-cuppable c.e. degree.

Theorem (Harrington)

1. *There is a high non-cuppable c.e. degree.*
2. *Moreover, for any high* $\mathbf{b}$ *there is a high* $\mathbf{a}$ *such that* $\mathbf{a}$ *cannot be cupped to* $\mathbf{b}$*:*

$$\forall \mathbf{x} \;\; \mathbf{a} \cup \mathbf{x} \geq \mathbf{b} \Rightarrow \mathbf{x} \geq \mathbf{b}.$$

A almost-everywhere dominating $\Rightarrow$ A is high...

so our result is a partial strengthening of Harrington's result (1).

An algebraic decomposition of c.e. degrees

A c.e. set A is either

An algebraic decomposition of c.e. degrees

A c.e. set A is either

- Cappable: $\exists$ c.e. B which computes nothing in common with A

$$W \leq_T A \text{ and } W \leq_T B \Rightarrow W \equiv_T \emptyset$$

the only things $\leq_T$ both A and B are the computable sets.
(aka minimal pair)

or

An algebraic decomposition of c.e. degrees

A c.e. set A is either

- Cappable: $\exists$ c.e. B which computes nothing in common with A

$$W \leq_T A \text{ and } W \leq_T B \Rightarrow W \equiv_T \emptyset$$

the only things $\leq_T$ both A and B are the computable sets.

(aka minimal pair)

or

- Promptly simple (definition omitted)

Cappables form an ideal; promptly simples a filter.

NCup is a subideal of cappables, due to

Theorem (Harrington Cup or Cap Theorem)

Every c.e. degree is either cuppable or cappable (or both).

Thus non-cuppable implies cappable.

Theorem (Barmpalias, Montalbán)

There is a cappable AED c.e. set.

Theorem (Barmpalias, Montalbán)

There is a cappable AED c.e. set.

A non-cuppable $\Rightarrow$ A cappable...

so our result is a strengthening of Barmpalias & Montalbán.

A corollary

As **NCup** is an ideal, we get an easy corollary:

Corollary

If there is a c.e. set $\leq_T$ all c.e. AED sets, then it must be non-cuppable.

It is not known if there is such a set - but it may be hard to construct.

Constructing a non-cuppable AED A

We make use of low-for-random reducibility:

$$A \leq_{LR} B \text{ iff all } B\text{-randoms are } A\text{-random.}$$

A, used as an oracle, is no better at detecting patterns than B.

Constructing a non-cuppable AED A

We make use of low-for-random reducibility:

$$A \leq_{LR} B \text{ iff all } B\text{-randoms are } A\text{-random.}$$

A, used as an oracle, is no better at detecting patterns than B.

Theorem (Kjos-Hanssen, Miller, Solomon)

A is AED iff $\emptyset' \leq_{LR} A$.

That is, A is *LR-complete* iff it is AED.

So instead of making A AED, we can make it $\geq_{LR} \emptyset'$.

How?

So instead of making A AED, we can make it $\geq_{LR} \emptyset'$.

How? Another theorem (Kjos-Hanssen):

Theorem (Kjos-Hanssen)

$$B \leq_{LR} A \text{ iff } U^B \subseteq V^A$$

for:

- U - member of universal oracle ML-test
- V^A - $\Sigma^0_1(A)$-class with $\mu(V^A) < 1$

So, to make $\emptyset' \leq_{LR} A$:

- if σ appears in $U^{\emptyset'}$, enumerate it into V^A with large use u
- if σ is removed from $U^{\emptyset'}$ due to $\emptyset'$-change, put u into A
- this may remove some other legitimate intervals ρ with use $r > u$; put ρ back into V^A with same use r.

Making A non-cuppable

To make A non-cuppable we would like to build Turing functional Δ to satisfy

$$N_e : \Gamma^{A \oplus W} = \emptyset' \Rightarrow \Delta^W = \emptyset'$$

for all Turing functionals Γ and c.e. sets W.

Idea:

- Wait until $\Gamma^{AW}(p)\downarrow = \emptyset'(p)$;
- define $\Delta^W(p) = \Gamma^{AW}(p)$;
- restrain $A \upharpoonright use\,\Gamma^{AW}(p)$.

Non-cupping strategy - naive

Problems:

1. if in fact $\Gamma^{AW} = \emptyset'$, we must act infinitely often

   $\Rightarrow$ N_e imposes infinite restraint

   $\Rightarrow$ must spread actions over infinitely many subrequirements

$$M_{e,p} : \Gamma^{AW}(p) = \emptyset'(p) \Rightarrow \Delta^{W}(p)\downarrow = \emptyset'(p)$$

2. need to be able to invalidate $\Delta^{W}(p)$ definitions to right of current path
   - must maintain A-restraint while $\Delta^{W}(p)$ is defined
   - need a way to force W-change

Non-cupping strategy - improved

We build auxiliary c.e. set D. Let

$$K = D \cup \emptyset' \qquad (\equiv_T \emptyset')$$

N Parent node: τ

- waits for expansionary stage for $\Gamma^{AW} = K$

M_p Subrequirement node: α

- chooses *flip-point* $d \notin D$
- waits until $\Gamma^{AW}(d)\downarrow$
- defines $\Delta^{W}(p)\downarrow = \Gamma^{AW}(p) = \emptyset'(p)$ with use $u = \mathit{use}\,\Gamma^{AW}(d)$

If we need to invalidate α's $\Delta^{W}(p)$ definition:

- enumerate d into D
- K changes, so $\Gamma^{AW} = K$ is destroyed
- if $\Gamma^{AW} = K$ then Γ^{AW} must change to restore agreement with K
- but A is restrained, so W must change below

$$use\,\Gamma^{AW}(d) = use\,\Delta^{W}(p)$$

- previous definition $\Delta^{\sigma}(p)$ is invalidated as now $\sigma \not\subset W$

Putting them together - non-cuppable and AED

- Restraints by non-cupping requirements prevent us from removing intervals from V^A
- Give each requirement a quota ϵ
- Allow it to capture at most ϵ junk intervals
- Choose ϵ's so that

$$\sum \epsilon < \frac{1}{2}$$

Thus

$$\mu(V^A) \ < \ \mu(U^{\emptyset'}) + \sum \epsilon \ < \ 1.$$

In tree setting, this means:

- allowing only one restraint on each level of the tree
- providing non-cupping requirements with an estimate to $\mu\left(U^{\emptyset'}\right)$
- resetting nodes if their measure estimate is wrong

(As in previous AED constructions)

Notable features of the construction

Regarding the AED strategy:

- Uses measure-guessing backup strategies as in previous AED constructions
- Can't always reset a node when its measure guess is wrong
  - use non-cupping clearing procedure instead
- Permanent restraints can capture more than their quota ϵ of junk intervals
- But still ensure that

$$\sum_{M_p} \epsilon(M_p) < 3\,\epsilon(N)$$

Notable features of the construction

Regarding the non-cuppable strategy:

- Must delay the definition of $\Delta^W(p)$ until

$$\mu\left(V^{A\upharpoonright u} - V^{A\upharpoonright R} - U^{\emptyset'}\right) < \epsilon$$

  That is, until we won't capture more than ϵ junk.
- Must clear definitions by nodes to the left, as well as above, before visiting a node

Further questions

Recall Harrington's theorem

Theorem

For all high c.e. sets B, there is a high c.e. A such that

$$A \oplus W \geq_T B \Rightarrow W \geq_T B, \ \forall \text{ c.e. } W.$$

We made A AED, for the case of $B = \emptyset'$.

Further questions

Recall Harrington's theorem

Theorem

For all high c.e. sets B, there is a high c.e. A such that

$$A \oplus W \geq_T B \Rightarrow W \geq_T B, \ \forall \text{ c.e. } W.$$

We made A AED, for the case of $B = \emptyset'$.

Can we make B *and* A AED?

Further questions

Can we make A even higher?

A is ultrahigh if $\emptyset'$ is strongly jump-traceable relative to A. Known that

$$A \text{ ultrahigh } \Rightarrow A \text{ AED.}$$

Further questions

Can we make A even higher?

A is ultrahigh if $\emptyset'$ is strongly jump-traceable relative to A. Known that

$$A \text{ ultrahigh } \Rightarrow A \text{ AED.}$$

Is there a non-cuppable ultrahigh c.e. set?